Estate taxes, consumption externalities, and altruism ☆

Jaime Alonso-Carrera [a], Jordi Caballé [b,*], Xavier Raurich [c]

[a] *Departamento de Fundamentos del Análisis Económico and RGEA, Universidade de Vigo, Fac, CC, Económicas e Empresariais, Campus As Lagoas-Marcosende, 36310 Vigo, Spain*
[b] *Unitat de Fonaments de l'Anàlisi Economica and CODE, Universitat Autònoma de Barcelona, Edifici B, 08193 Barcelona, Spain*
[c] *Departament de Teoria Econòmica and CREB, Universitat de Barcelona, Facultat CC, Econòmiques i Empresarials, Av, Diagonal 690, 08034 Barcelona, Spain*

A R T I C L E   I N F O

A B S T R A C T

We study how the introduction of consumption externalities affects the optimality of the dynamic equilibrium in an economy displaying dynastic altruism. When the bequest motive is inoperative consumption externalities affect the intertemporal margin between young and old consumption and thus modify the intertemporal path of aggregate consumption and capital. The optimal tax policy that solves this intertemporal suboptimality consists of a tax on capital income and a pay-as-you-go social security system. The latter solves the excess of capital accumulation due to the inoperativeness of the bequest motive and the former solves the suboptimal allocation of consumption due to consumption externalities. When the bequest motive is operative consumption externalities only cause an intratemporal misallocation of consumption but do not affect the optimality of the capital stock level. This suboptimal allocation of consumption implies in turn that the path of bequest deviates also from optimality. The optimal tax policy in this case consists of an estate tax and a capital income tax.

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1. Introduction

In this paper we aim at analyzing the suboptimal allocation arising from consumption externalities in an overlapping generation model (OLG) with dynastic altruism. We use this model to show how the interaction between consumption externalities and altruism affects the optimality of the equilibrium path. In our model the consumption externality will take the form of a reference level of consumption that is used to compare the utility derived from own consumption. We will assume that this reference consumption is a weighted average of the consumption of all the agents living in the same period.

Several authors have analyzed the implications of a consumption reference arising from own past consumption (or internal habits) in OLG models. Examples of this strand of the literature are the papers of Lahiri and Puhakka (1998) and Wendner (2002), who study the effect of habits on saving in a pure exchange economy; and Alonso-Carrera et al. (2007), de la Croix (1996) and de la Croix and Michel (1999, 2001), who analyze several related stability issues. Obviously, when the reference is the others' consumption, then suboptimality of the equilibrium path is likely to arise. In a framework with infinitely-lived agents, Alonso-Carrera et al. (2004 and 2005), Fisher and Hof (2000), Guo (2005), Liu and Turnovsky (2005), Ljungqvist and Uhlig (2000), and Turnovsky and Monteiro (2007), among many others, have characterized the optimal tax rates that solve this kind of suboptimality. In the framework of OLG models, Abel (2005) shows that a capital income tax and a pay-as-you-go social security system constitute the optimal tax policy. We extend the latter paper by introducing consumption externalities in the model with dynastic altruism of

Barro (1974). Following Abel's analysis, we assume that these externalities take the form of a weighted average of the consumption of the two types of agents living in the same period. By introducing altruism and the possibility of bequests in Abel's model we can analyze new phenomena, like the potential suboptimality of the level of bequests and the interaction between consumption externalities and the lack of optimality of the bequest constrained equilibrium.

The analysis of optimality in Abel (2005) is based on the comparison between the model of Diamond (1965) with consumption externalities and the corresponding planner problem when the objective function of the planner is just the discounted sum of the utilities of all the agents in the economy. This analysis does not allow to distinguish the deviation from optimality due to consumption spillovers from the one due to the inoperativeness of the bequest motive. In contrast, the analysis in our paper is based on the comparison between the model of Barro (1974) with consumption externalities and the corresponding planner problem. As shown by Abel (2005), when the bequest motive is inoperative, suboptimality is due to both consumption externalities and the typical capital overaccumulation problem (Cass, 1972). However, when the bequest motive is operative consumption externalities are the only source of suboptimality. Therefore, our analysis shows how the lack of optimality due to consumption externalities depends on the operativeness of the bequest motive.

When the bequest motive is operative consumption externalities only cause an intratemporal misallocation of resources, whereas they cause an intertemporal misallocation when the bequest motive is inoperative. In the former case, they only affect the intratemporal margin between consumption of the two generations living in the same period. This implies that they modify the allocation of consumption between the two living generations, but do not affect the optimality of the intertemporal paths of saving and output. In fact, this suboptimal allocation of consumption is associated with a suboptimal level of bequest. In contrast, when the bequest motive is inoperative, consumption externalities affect the intertemporal margin of consumption along the life cycle of agents. Given that consumption spillovers affect the intertemporal path of aggregate consumption, they also modify the paths of saving and production. The externality associated with young consumption reduces the stock of capital and, hence, reduces the overaccumulation of capital due to the inoperativeness of the bequest motive. Obviously, the externality associated with old consumption has the opposite effect on the stock of capital and thus increases the gap between the optimal stock of capital and the one obtained in a competitive equilibrium.

Note that the nature of suboptimality crucially depends on the operativeness of the bequest motive and this operativeness depends in turn on the intensity of consumption externalities. In this respect we show that in the long run a stronger young consumption reference reduces the critical level of altruism above which the bequest motive is operative, whereas a larger old consumption reference rises this critical level of altruism.

The optimal values of tax rates also depend on the operativeness of the bequest motive. When the bequest motive is inoperative, the optimal tax policy consists of a tax on capital income and a pay-as-you-go social security system. The latter solves the excess of capital accumulation and the former solves the suboptimal allocation of consumption due to consumption externalities. Note that the optimal capital income tax rate modifies the intertemporal path of aggregate consumption and, thus, solves the lack of optimality due to consumption spillovers. When the bequest motive is operative, the optimal tax policy consists of an estate tax and a capital income tax. Moreover, in this case the pay-as-you-go social security system is not required as there is no overaccumulation of capital.

The paper is organized as follows. Section 2 presents the model. Section 3 characterizes the competitive equilibrium when the bequest motive is operative and when it is not. Section 4 characterizes the social planner solution. In Section 5 we conduct the analysis of optimality by comparing the solution achieved by the planner with the competitive solution. Section 6 characterizes the optimal tax rates. Section 7 concludes the paper. All the proofs appear in the Appendix A.

2. The model

Let us consider an OLG model where N_t identical individuals are born in period t. These individuals live for two periods. Each individual has an offspring at the end of the first period of his life and the number of children per parent is $n \geq 1$. As in Diamond (1965), each agent supplies inelastically one unit of labor in the first period of his life and is retired in the last period of his life. We index each generation by the period in which its members work.

Individuals are assumed to be altruistic towards their children. Let b_t be the amount of bequest that an old individual leaves to each of their children in period t. We impose the constraint that parents cannot force their descendents to give them gifts,

$$b_t \geq 0. \tag{1}$$

Each young individual distributes his labor income and his inheritance between consumption and saving. Therefore, the budget constraint faced by an individual during his first period of life is

$$w_t + b_t = c_t + s_t, \tag{2}$$

where c_t is the amount of consumption of a young agent, w_t is the labor income and s_t is the amount saved. In the second period of life individuals receive a return on the amount of their saving, which is distributed between consumption and bequests for their children. Therefore, the budget constraint of an old individual is

$$R_{t+1}s_t = x_{t+1} + nb_{t+1}, \tag{3}$$

where R_{t+1} is the gross rate of return on saving and x_{t+1} is the amount of consumption of an old individual.

The utility function of an individual belonging to generation t is

$$V_t = U(\hat{c}_t, \hat{x}_{t+1}) + \beta V_{t+1}, \quad \text{for} \quad t = 0, 1, 2, \ldots \tag{4}$$

where V_{t+1} represents the indirect utility of each of his descendants, the parameter $\beta \in [0,1)$ is the altruism factor,[1] and the variables $\hat{c}_t$ and $\hat{x}_{t+1}$ represent the effective consumption in the first and second periods of life, respectively, of a representative individual belonging to generation t. We assume that individuals do not derive utility from the absolute level of consumption but from the comparison between their consumption and some consumption reference. In particular, we assume the following functional forms for effective consumption:

$$\hat{c}_t = c_t - \gamma v_t^y, \tag{5}$$

and

$$\hat{x}_{t+1} = x_{t+1} - \delta v_{t+1}^o, \tag{6}$$

where $\gamma \in [0,1)$ and $\delta \in [0,1)$ provide a measure of the intensity of the consumption reference.[2] These consumption references are assumed to be a weighted arithmetic average of the per capita consumption of the two living generations. On the one hand, we assume that

$$v_t^y = \frac{N_t c_t + \theta^y N_{t-1} x_t}{N_t + \theta^y N_{t-1}} = \left(\frac{n}{n + \theta^y}\right) c_t + \left(\frac{\theta^y}{n + \theta^y}\right) x_t, \tag{7}$$

where $\theta^y \in [0,1]$ is the weight of consumption of a representative old consumer in the specification of the reference for young consumers. On the other hand, we assume that

$$v_{t+1}^o = \frac{\theta^o N_{t+1} c_{t+1} + N_t x_{t+1}}{\theta^o N_{t+1} + N_t} = \left(\frac{\theta^o n}{\theta^o n + 1}\right) c_{t+1} + \left(\frac{1}{\theta^o n + 1}\right) x_{t+1}, \tag{8}$$

where $\theta^o \in [0,1]$ is the weight of consumption of a representative young consumer in the specification of the reference for old consumers. Note that the restrictions imposed on the values of the parameters θ^y and θ^o imply that we are giving a larger weight to the average consumption of the agents belonging to the same generation. Let us define $\varepsilon^y = \frac{n}{n+\theta^y}$ and $\varepsilon^o = \frac{\theta^o n}{\theta^o n+1}$. Then, Eqs. (7) and (8) can be rewritten as follows:

$$v_t^y = \varepsilon^y c_t + (1 - \varepsilon^y) x_t, \tag{9}$$

and

$$v_{t+1}^o = \varepsilon^o c_{t+1} + (1 - \varepsilon^o) x_{t+1}. \tag{10}$$

As in Abel (1986) or Laitner (1988), we assume that the function $U(\cdot,\cdot)$ is twice continuously differentiable and additive in its two arguments. Therefore, we will use the following functional form:

$$U(\hat{c}_t, \hat{x}_{t+1}) = u(\hat{c}_t) + \rho u(\hat{x}_{t+1}), \tag{11}$$

where $\rho > 0$ is the temporal discount factor. We assume that $u'(z) > 0$, $u''(z) < 0$ for $z > 0$ and the Inada conditions $\lim_{z \to 0} u'(z) = \infty$ and $\lim_{z \to \infty} u'(z) = 0$.

In period 0 there is an old generation of N_{-1} individuals born in period -1. When they are old, the individuals of generation -1 have preferences represented by the utility function:

$$\rho u(\hat{x}_0) + \beta V_0. \tag{12}$$

There is a single commodity in this economy, which can be devoted to either consumption or investment. Let us assume that this commodity is produced by means of a neoclassical production function $\hat{F}(K_t, L_t)$, where K_t is the capital stock and L_t is the amount of labor used in period t. Capital depreciates every period at the rate $v \in [0,1]$. The production function per capita is $\hat{f}(k_t)$, where k_t is the capital stock per capita. We assume that $\hat{f}'(k) > 0$, $\hat{f}''(k) < 0$ for all $k > 0$ and the Inada conditions $\lim_{k \to 0} \hat{f}'(k) = \infty$ and $\lim_{k \to \infty} \hat{f}'(k) = 0$. To ease the notation, we introduce the function $f(k_t) = \hat{f}(k_t) + (1-v)k_t$ and, hence, $\lim_{k \to 0} f'(k) = \infty$ and $\lim_{k \to \infty} f'(k) = 1 - v$. As firms behave competitively, the rental prices of the two inputs equal their marginal productivities

$$R_t = f'(k_t) \equiv R(k_t), \tag{13}$$

$$w_t = f(k_t) - f'(k_t) k_t = \hat{f}(k_t) - \hat{f}'(k_t) k_t \equiv w(k_t). \tag{14}$$

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[1] Each parent cares equally about the felicity of their n children. Thus, the intercohort utility discount β could be rewritten as $\beta = n\rho\beta'$, where ρ would be the temporal discount factor and β' is the pure interpersonal (from parents to children) discount factor.

[2] We assume an additive specification for effective consumption instead of the multiplicative formulation of Abel (2005) in order to guarantee concavity of the social planner's utility function (see Alonso-Carrera et al., 2005).

In equilibrium the capital stock installed in period $t+1$ is equal to the aggregate saving in period t and, thus, we have

$$nk_{t+1} = s_t, \quad \text{for} \quad t = 0, 1, 2, \dots . \tag{15}$$

3. Competitive equilibrium

On the one hand, the problem faced by each individual belonging to generation t, $t=0, 1, 2,\dots$ is to maximize (4) with respect to $\{c_t, x_{t+1}, b_{t+1}\}$ subject to (1), (2), (3), (5) and (6), which is equivalent to solving the following dynamic programming problem:

$$V_t(b_t) = \max_{\{s_t, b_{t+1}\}} \{u(w_t + b_t - s_t - \gamma v_t^y) + \rho u(R_{t+1}s_t - nb_{t+1} - \delta v_{t+1}^o) + \beta V_{t+1}(b_{t+1})\}, \tag{16}$$

with $b_{t+1} \geq 0$, for v_t^y, v_{t+1}^o, w_t and R_{t+1} given for all t. On the other hand, the problem faced by the individuals belonging to generation -1 when they are old (i.e., in period 0) is equivalent to the following:

$$\max_{\{b_0\}} \{\rho u(R_0 k_0 - nb_0 - \delta v_0^o) + \beta V_0(b_0)\}, \tag{17}$$

with $b_0 \geq 0$, for k_0, v_0^o, and R_0 given. Note that k_0 is the endowment of capital of an old individual in period 0.

Using the envelope theorem in Eq. (16) we obtain,

$$\frac{\partial V_t}{\partial b_t} = u'(\hat{c}_t), \quad \text{for} \quad t = 0, 1, 2, \dots . \tag{18}$$

Using (18), we obtain the first order conditions for problem (16) corresponding to the derivative with respect to s_t,

$$u'(c_t - \gamma v_t^y) = \rho R_{t+1} u'(x_{t+1} - \delta v_{t+1}^o), \quad \text{for} \quad t = 0, 1, 2, \dots . \tag{19}$$

Moreover, the first order condition for problems (17) and (16) with respect to bequests b_t is

$$n\rho u'(x_t - \delta v_t^o) \geq \beta u'(c_t - \gamma v_t^y), \quad \text{for} \quad t = 0, 1, 2, \dots , \tag{20}$$

where the condition holds with equality if $b_t > 0$. Eq. (19) characterizes the optimal allocation of consumption along the lifetime of an individual. If the bequest motive is operative, then Eq. (20) characterizes the optimal allocation of consumption between the two consecutive generations $t-1$ and t. This equation tells us that, when the bequest motive is operative ($b_t > 0$), the utility loss of parents arising from a larger amount of bequest must be equal to the discounted utility gain of their direct descendants.

The competitive equilibrium of this economy is a path $\{k_{t+1}, c_t, x_t, b_t\}_{t=0}^{\infty}$ that, for a given initial value of k_0, solves the system of difference equations composed of (19) and (20), together with (1), (2), (3), (9), (10), (13), (14), (15) and the transversality condition

$$\lim_{t \to \infty} \beta^t u'(c_t) b_t = 0. \tag{21}$$

The previous transversality condition states that the present value of the long run amount of bequests tends to zero.

We will restrict our analysis to steady state equilibria, that is, competitive equilibria where the variables k_t, c_t, x_t and b_t are all constant.[3] To this end, we combine (19) with (9) and (10) to obtain

$$u'[(1 - \gamma \varepsilon^y)c - \gamma(1 - \varepsilon^y)x] - \rho R u'[(1 - \delta(1 - \varepsilon^o))x - \delta \varepsilon^o c] = 0,$$

which after using (2), (3), (13), (14) and (15) becomes

$$h(k, b) \equiv u'[\hat{c}(k, b)] - \rho R(k) u'[\hat{x}(k, b)] = 0 \tag{22}$$

with

$$\hat{c}(k, b) = (1 - \gamma \varepsilon^y)(w(k) + b - nk) - \gamma(1 - \varepsilon^y)n(f'(k)k - b)$$

and

$$\hat{x}(k, b) = (1 - \delta(1 - \varepsilon^o))n(f'(k)k - b) - \delta \varepsilon^o(w(k) + b - nk).$$

The function $h(k, b)$ is only defined for the values $k > 0$ and $b \geq 0$ for which $\hat{c}(k, b) > 0$ and $\hat{x}(k, b) > 0$.

The following two lemmas provide two useful properties of the function $h(k, b)$. The proofs of the two lemmas follow directly from a direct computation of the corresponding partial derivatives of $h(k, b)$ and the competitive rental prices (13) and (14). The next lemma refers to the behavior of the function h when the stationary amount of bequests varies:

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[3] We will suppress the time subindex when we refer to the steady state equilibrium value of a variable.

Lemma 3.1. $h_b \equiv \frac{\partial h}{\partial b} < 0$ for all $b > 0$.

The next lemma provides a sufficient condition under which the function $h(k, b)$ is increasing in the capital stock per capita for all $b \geq 0$:

Lemma 3.2. *Assume that the following condition holds:*

$$|f''(k)| < \min\{n/k, f'(k)/k\}. \tag{23}$$

Then, $h_k \equiv \frac{\partial h}{\partial k} > 0$ for all $b \geq 0$.

Note that condition (23) can be written as

$$f''(k)k + n > 0, \tag{24}$$

and

$$f''(k)k + f'(k) > 0. \tag{25}$$

Clearly, the inequality (24) implies that the stationary first period consumption c is decreasing in the stationary capital stock, whereas the inequality (25) implies that the stationary second period consumption x is increasing in the stationary capital stock at a steady state equilibrium.

As in Abel (1987), we are going to introduce the assumption that the steady state equilibrium value $\bar{k}$ of capital in an economy with no bequests is unique and compatible with positive effective consumptions and that $h(k, 0) \gtreqless 0$ whenever $k \gtreqless \bar{k}$. Moreover, we will make the generic assumption that the slope of $h(k, 0)$ at $\bar{k}$, $h_k(\bar{k}, 0)$, is different from zero. Obviously, Lemma 3.2 provides a sufficient condition under which this assumption holds. Moreover, notice that these assumptions are immediately satisfied when u is logarithmic, $\hat{F}$ is Cobb–Douglas and the values of the parameters δ and γ are sufficiently small.

Assumption A. There exists a unique strictly positive value $\bar{k}$ satisfying $h(k, 0) = 0$ with $\hat{c}(\bar{k}, 0) > 0$ and $\hat{x}(\bar{k}, 0) > 0$. Moreover, $h_k(\bar{k}, 0) > 0$.

Let us combine conditions (19), (20) when it is just binding, and (13), all of them evaluated at the steady state with no bequests, to obtain the threshold value of the altruism factor β above which bequests are positive,

$$\bar{\beta} = \frac{n}{f'(\bar{k})}. \tag{26}$$

Note that this threshold value $\bar{\beta}$ coincides with the one obtained in Weil (1987). As we will see, consumption externalities will modify the value of $\bar{\beta}$ through their effect on the value $\bar{k}$ of the stationary capital stock.

Proposition 3.3. *a) If* Assumption A *holds and $\beta \leq \bar{\beta}$ then the unique steady state equilibrium satisfies the following equations:*

$$\bar{b} = 0,$$

$$h(\bar{k}, 0) = 0,$$

$$\bar{c} = f(\bar{k}) - \bar{k}f'(\bar{k}) - n\bar{k},$$

and

$$\bar{x} = nf'(\bar{k})\bar{k},$$

where $\bar{k}, \bar{b}, \bar{c}$ and $\bar{x}$ are the steady state values of capital, bequests and consumption when young and when old, respectively.
b) If $\beta > \bar{\beta}$ then the steady state is unique, exhibits a strictly positive amount of bequests and satisfies the following equations:

$$f'(k^*) = \frac{n}{\beta}, \tag{27}$$

$$h(k^*, b^*) = 0,$$

$$c^* = f(k^*) - k^*f'(k^*) - nk^* + b^*,$$

and

$$x^* = nf'(k^*)k^* - nb^*,$$

where k^*, b^*, c^* and x^* are the steady state values of capital, bequest and consumption when young and when old, respectively.

Observe that the capital stock when the bequest motive is operative is implicitly given in Eq. (27) and is thus the one associated with the modified golden rule. However, the capital stock when the bequest motive is inoperative is smaller than the one corresponding to that rule.

Proposition 3.4. *a) Let* Assumption A *and condition* (25) *hold. Then, the steady state equilibrium with inoperative bequest motive is locally saddle path stable for values of δ and γ sufficiently close to zero. b) The steady state equilibrium with operative bequest motive is locally saddle path stable.*

Whereas the steady state with positive bequests is always locally saddle path stable, the local saddle path stability of the steady state with zero bequests can be obtained under Assumption A, condition (25), which means that second period consumption is locally increasing in the stationary capital stock, and when the intensity of consumption externalities, which is summarized by the value of the parameters δ and γ, is small. However, as is shown in the Appendix A, when the intensity of consumption externalities is sufficiently large, the steady state with zero bequests may not be saddle path stable under these conditions.[4] For the sake of completeness, we show in Table 1 of Appendix B how the stability properties of a steady state with inoperative bequest motive vary with the value of the parameters δ and γ.

The next proposition provides some comparative statics results when the intensity of consumption externalities varies:

Proposition 3.5. *a) If $\beta > \bar{\beta}$, then $\frac{\partial k^*}{\partial \gamma} = 0$, $\frac{\partial k^*}{\partial \delta} = 0$, $\frac{\partial b^*}{\partial \gamma} > 0$, $\frac{\partial b^*}{\partial \delta} < 0$. b) If* Assumption A *holds and $\beta < \bar{\beta}$, then $\frac{\partial \bar{k}}{\partial \delta} > 0$, and $\frac{\partial \bar{k}}{\partial \gamma} < 0$.*

The previous proposition tells us in its part (a) that, if the amount of bequest is positive, then consumption externalities modify the intergenerational distribution of consumption but do not affect the long run value of capital. In other words, consumption externalities neither modify the amount of saving nor the output level, but they modify the allocation of consumption between young and old generations. This change in the allocation of consumption is achieved by adjusting the amount of bequest. An increase in the value of the parameter γ raises the marginal valuation of young consumption and, as follows from (20), this results in a utility gain from a larger amount of inheritances. In contrast, as δ increases agents are willing to increase old consumption and this requires a reduction in the amount of bequest.

An increase in γ induces agents to increase young consumption, whereas an increase in δ induces agents to increase old consumption. Accordingly, part (b) of Proposition 3.5 tells us that, when the equilibrium amount of bequests is zero, agents rise young (old) consumption by decreasing (increasing) the amount of saving. This explains the effect of δ and γ on the capital stock since savings coincide with the stock of capital in equilibrium.

We next show how the operativeness of the bequest motive is affected by consumption spillovers:

Proposition 3.6. $\frac{\partial \bar{\beta}}{\partial \delta} > 0$ *and* $\frac{\partial \bar{\beta}}{\partial \gamma} < 0$.

According to the previous proposition, if young individuals become more sensitive to the others' consumption (γ increases), positive bequests are more likely to appear in a stationary equilibrium. The converse occurs when the effect of consumption spillovers on old consumers becomes stronger (δ increases). Clearly, these results show that the introduction of consumption externalities affects the operativeness of the bequest motive and, thus, affects the dynamic optimality of the equilibrium path (Cass, 1972).

4. The social planner solution

We assume that the social planner gives the same weight to all the individuals belonging to the same generation in his objective function and the intergenerational discount factor coincides with private altruistic factor β. Therefore, the social planner maximizes[5]

$$\hat{U}_t = \beta^{-1}[\rho u((1 - \delta(1 - \varepsilon^o))x_0 - \delta\varepsilon^o c_0)] + \sum_{t=0}^{\infty} \beta^t [u((1 - \gamma\varepsilon^y)c_t - \gamma(1 - \varepsilon^y)x_t) + \rho u((1 - \delta(1 - \varepsilon^o))x_{t+1} - \delta\varepsilon^o c_{t+1})], \tag{28}$$

subject to the resource constraint

$$f(k_t) = c_t + \frac{x_t}{n} + nk_{t+1}, \quad \text{for} \quad t = 0, 1, 2, \dots \tag{29}$$

with k_0 given. Note that the first term in the expression for $\hat{U}_t$ in Eq. (28) refers to the old generation in period 0. Thus, the Lagrangean of the planner's maximization problem can be written as

$$\mathcal{L} = \beta^{-1}[\rho u((1 - \delta(1 - \varepsilon^o))x_0 - \delta\varepsilon^o c_0)] + \sum_{t=0}^{\infty} \beta^t [u((1 - \gamma\varepsilon^y)c_t - \gamma(1 - \varepsilon^y)x_t) + \rho u((1 - \delta(1 - \varepsilon^o))x_{t+1} - \delta\varepsilon^o c_{t+1})]$$

$$+ \sum_{t=0}^{\infty} \lambda_t \left(f(k_t) - c_t - \frac{x_t}{n} - nk_{t+1} \right).$$

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[4] In related OLG models where individual preferences are not subject to consumption externalities, de la Croix and Michel (2001) and Alonso-Carrera et al. (2007) show that the steady state with inoperative bequest is saddle path stable when $h_k > 0$. However, this condition is not sufficient to guarantee saddle path stability when consumption externalities are present.

[5] Note that this objective function coincides with the planner's objective function in Abel (2005).

The corresponding first order conditions with respect to c_t, x_t, and k_{t+1} are

$$(1 - \gamma \varepsilon^y)\beta^t u'\left(\hat{c}_t^p\right) - \delta \varepsilon^o \beta^{t-1} \rho u'\left(x_t^p\right) - \lambda_t = 0, \tag{30}$$

$$-\gamma(1 - \varepsilon^y)\beta^t u'\left(\hat{c}_t^p\right) + [1 - \delta(1 - \varepsilon^o)]\beta^{t-1}\rho u'\left(\hat{x}_t^p\right) - \frac{\lambda_t}{n} = 0, \tag{31}$$

and

$$\lambda_{t+1} f'\left(k_{t+1}^p\right) - n\lambda_t = 0, \tag{32}$$

for $t = 0, 1, 2,\ldots$, where the superscript p denotes an optimal path from the planner's viewpoint. Combining (30) and (31), we obtain

$$\left(\frac{u'\left(\hat{x}_t^p\right)}{u'\left(\hat{c}_t^p\right)}\right)\left(\frac{\rho}{I}\right) = \frac{\beta}{n}, \tag{33}$$

where

$$I \equiv \frac{1 - \gamma((1 + n)\varepsilon^y - n)}{1 - \delta\left(1 - \left(1 + \frac{1}{n}\right)\varepsilon^o\right)}.$$

Using (30), (31) and (32), we get

$$\left(\frac{u'\left(\hat{c}_{t+1}^p\right)}{u'\left(\hat{c}_t^p\right)}\right)\left(\frac{\beta}{n}\right) = \frac{1}{f'\left(k_{t+1}^p\right)}. \tag{34}$$

The social planner solution is a path $\left\{k_{t+1}^p, c_t^p, x_t^p\right\}_{t=0}^{\infty}$ that, for a given initial value of k_0^p, solves the system of difference Eqs. (29), (33) and (34), together with the transversality condition

$$\lim_{t \to \infty}\left[\beta^t u'\left(\hat{c}_t^p\right)(1 - \gamma \varepsilon^y) - \beta^{t-1}\rho u'\left(\hat{x}_t^p\right)\delta \varepsilon^o\right]k_{t+1} = 0.$$

Proposition 4.1. *There exists a unique steady state in the social planner problem. This steady state is given by*

$$f'(k^p) = \frac{n}{\beta}, \tag{35}$$

$$\frac{u'\left(\hat{x}^p\right)}{u'\left(\hat{c}^p\right)} = \frac{I\beta}{n\rho}, \tag{36}$$

and

$$f(k^p) = c^p + \frac{x^p}{n} + nk^p, \tag{37}$$

where k^p, c^p and x^p are the steady state values of capital, consumption when young and when old, respectively.

The following proposition characterizes the dynamics around the steady state of the planner's solution and its proof is omitted since is similar to that of part (b) of Proposition 3.4:

Proposition 4.2. *The steady state of the social planner solution is locally saddle path stable.*

5. Planner optimality

There are two different sources of deviations from the planner's optimal solution in our economy: the contemporaneous consumption externalities and the inoperativeness of the bequest motive. In what follows, we will analyze the effects of these different sources of suboptimality, which interact in a non-obvious way. To this end, we first compare the social planner solution with the competitive equilibrium solution when bequests are positive, and then we perform the same comparison when bequests are zero.

5.1. Suboptimality of the equilibrium path when the bequest motive is operative

When the bequest motive is operative, consumption externalities are the only source of deviation from planner optimality. Note that the demographic structure of our OLG model makes this case different from the model with an infinitely-lived representative agent, which we considered in Alonso-Carrera et al. (2004). When the individuals of the same family are effectively linked through positive bequests, the path of aggregate consumption in a given period is optimal in spite of contemporaneous externalities. However, the existence of contemporaneous consumption externalities when there are agents with different ages living in the same period makes suboptimal the intratemporal allocation of consumption between the two generations.[6]

We can compare the social planner solution, which is characterized by Eqs. (29), (33) and (34), with the competitive equilibrium solution when the bequest motive is operative, which is characterized by Eqs. (19), (20) and (29). We first rewrite the equilibrium solution by rearranging (20) as follows:

$$\left(\frac{u'(\hat{x}_t)}{u'(\hat{c}_t)}\right)\rho = \frac{\beta}{n}, \tag{38}$$

where the left hand side (LHS) is the private marginal rate of substitution (MRS) between young and old consumption at period t and the right hand side (RHS) is the private marginal rate of transformation (MRT). Thus, Eq. (38) determines the intratemporal allocation of consumption between agents belonging to different generations along the equilibrium path. Combining (19) with (20), we obtain

$$\left(\frac{u'(\hat{c}_{t+1})}{u'(\hat{c}_t)}\right)\left(\frac{\beta}{n}\right) = \frac{1}{f'(k_{t+1})}, \tag{39}$$

where the LHS is the private MRS between young and old consumption of individuals belonging to generation t and the RHS is the corresponding private MRT. It then follows that Eq. (39) determines the intertemporal allocation of consumption along the equilibrium path.

We follow a similar reasoning with the social planner solution. On the one hand, (33) determines the optimal intratemporal allocation of consumption between agents belonging to different generations. On the other hand, (34) determines the optimal intertemporal allocation of consumption. This equation is the Keynes–Ramsey equation that, given an initial condition on the stock of capital, characterizes the optimal path of saving and thus the optimal path of capital stock and output.

From the comparison of (33) and (34) with (38) and (39) we see that consumption externalities affect the intratemporal allocation of consumption between generations, as the private and social MRS do not coincide, but do not affect the intertemporal allocation of consumption. Therefore, consumption externalities do not affect the optimality of the capital path when the bequest motive is operative. Given that production and savings are at its optimal level, this suboptimal allocation of consumption between young and old agents is associated with a suboptimal path of bequests. The size of the difference between the private and social MRS is given by the value of the variable I, which provides a measure of the inefficiency due to consumption externalities. Thus, the amount of bequest is suboptimally small (large) when the externalities make agents value old (young) consumption in excess to young (old) consumption. In fact, it can be shown that bequests are suboptimally small (large) whenever $I>(<)1$. Note also that the two consumption externalities result in opposite effects and, in fact, when $I=1$ the competitive equilibrium is efficient even though consumption externalities are present.

5.2. Suboptimality of the equilibrium path when the bequest motive is inoperative

When the amount of bequest is zero, the equilibrium is characterized by Eqs. (3), (19) and (29). We proceed to compare these equations with those characterizing the social planner solution. First, we combine (33) with (34) to obtain

$$\left(\frac{u'\left(\hat{x}^p_{t+1}\right)}{u'\left(\hat{c}^p_t\right)}\right)\left(\frac{\rho}{I}\right) = \frac{1}{f'\left(k^p_{t+1}\right)}, \tag{40}$$

where the LHS is the social MRS between consumption when young and when old and the RHS is the corresponding social MRT. We can compare this equation with the equilibrium Eq. (19), which can be rewritten as

$$\left(\frac{u'(\hat{x}_{t+1})}{u'(\hat{c}_t)}\right)\rho = \frac{1}{f'(k_{t+1})},$$

---

[6] In Alonso-Carrera et al. (2004) there is no heterogeneity among agents so that the optimality of the aggregate consumption path implies an optimal intratemporal allocation of consumption. We also prove in that paper that contemporaneous consumption externalities give rise to suboptimal consumption paths only if they interact with the consumption of previous periods.

where the LHS is the private MRS and the RHS is the corresponding private MRT. This equation drives the intertemporal margin between consumption when young and when old along an equilibrium path with inoperative bequest motive. Note that this equilibrium margin differs from the optimal margin in both the MRS and in the MRT. The difference between the social and the private MRS is due to consumption externalities and the difference between the social and the private MRT arises because the competitive stock of capital differs from its planned counterpart when the bequest motive is not operative. To see this, note that (34) does not hold along an equilibrium path when bequests are zero. Moreover, as follows from (3) and (19), consumption spillovers modify the equilibrium stock of capital and, hence, the private MRT when bequests are zero. Thus, when the bequest motive is not operative, consumption externalities affect both the MRS and the MRT and thus give rise to both a suboptimal level of production and a suboptimal allocation of consumption.

The long run value of the stock of capital is not optimal in this case. The optimal long run value of the stock of capital k^p is obtained from (34), whereas the long run value $\bar{k}$ of the stock of capital in the competitive equilibrium comes from the equation $h(\bar{k}, 0) = 0$. As follows from the condition (26) for the operativeness of the bequest motive, $\bar{k} > k^p$. Thus, the inoperativeness of the bequest motive implies a suboptimally large capital stock. The stock of capital in a steady state equilibrium can be expressed as a function of γ and δ, $\bar{k}(\gamma, \delta)$ whereas the optimal stock of capital can be written as a function of the altruism factor, $k^p(\beta)$. The overall inefficiency could be measured by the difference $\bar{k}(\gamma, \delta) - k^p(\beta)$. This inefficiency can be divided into two components: (i) the inefficiency due to the inoperativeness of the bequest motive; and (ii) the inefficiency due to consumption externalities. The former inefficiency is given by $\bar{k}(0, 0) - k^p(\beta)$. Since the strength of altruism rises the optimal stock of capital (see (35)) but does not affect the equilibrium stock of capital when the bequest motive is inoperative, this inefficiency decreases as β increases. On the other hand, the inefficiency due to consumption externalities is given by $\bar{k}(\gamma, \delta) - \bar{k}(0, 0)$. From Proposition 3.5, we can conclude that the size of this inefficiency decreases with γ and increases with δ.

6. Optimal taxes

In this section we proceed to characterize the optimal tax rates to implement an optimal allocation when consumption externalities interact with altruism. We consider a tax policy consisting of an estate tax, a capital income tax and a system of lump-sum taxes. Concerning the lump-sum taxes, we assume that young agents pay a lump-sum tax τ_t^y and the revenues are devoted to finance a lump-sum subsidy to the old $-\tau_t^o$. Therefore, these two tax rates are related by the following balanced budget constraint:

$$\tau_t^o = -n\tau_t^y. \tag{41}$$

We also assume that young agents pay an estate tax on the inheritance they receive and that the revenues accruing from these taxes are returned to the same young agents by means of a lump-sum subsidy. Thus, there is a second government budget constraint, which is

$$\phi_t^y = \tau_t^b b_t, \tag{42}$$

where ϕ_t^y is a lump-sum subsidy and τ_t^b is the estate tax rate. Finally, we assume that old agents pay a capital income tax on the returns of savings and that these revenues are returned to these agents by means of a lump-sum subsidy. Therefore, another government budget constraint is

$$\phi_t^o = \tau_t^k s_t f'(k_{t+1}), \tag{43}$$

where ϕ_t^o is a lump-sum subsidy and τ_t^k is the capital income tax rate.

Consider an individual belonging to generation $t = 0, 1, 2,\ldots$ who maximizes (4) with respect to $\{c_t, x_{t+1}, b_{t+1}\}$ subject to (1) and the following constraints:

$$c_t = w_t - \tau_t^y + \phi_t^y + \left(1 - \tau_t^b\right)b_t - s_t, \tag{44}$$

$$x_{t+1} = \left(1 - \tau_{t+1}^k\right)R_{t+1}s_t + \phi_{t+1}^o - nb_{t+1} - \tau_{t+1}^o, \tag{45}$$

which amounts to solve the following dynamic programming problem:

$$V_t(b_t) = \max_{\{s_t, b_{t+1}\}} \{u\left(w_t - \tau_t^y + \phi_t^y + \left(1 - \tau_t^b\right)b_t - s_t - \gamma v_t^y\right) + \rho u\left(\left(1 - \tau_{t+1}^k\right)R_{t+1}s_t + \phi_{t+1}^o - nb_{t+1} - \tau_{t+1}^o - \delta v_{t+1}^o\right) + \beta V_{t+1}(b_{t+1})\},$$

with $b_{t+1} \geq 0$, for v_t^y, v_{t+1}^o, w_t and R_{t+1} given for $t = 0, 1, 2,\ldots$.

The problem faced by the individuals belonging to generation -1 in period 0 is the following:

$$\max_{\{b_0\}} \left\{\rho u\left(\left(1 - \tau_0^k\right)R_0 k_0 + \phi_0^o - nb_{0 -}\tau_0^o - \delta v_0^o\right) + \beta V_0(b_0)\right\},$$

with $b_0 \geq 0$, for k_0, v_0^o, and R_0 given.

The first order conditions of the previous two problems with respect to s_t and b_t are

$$u'(c_t - \gamma v_t^y) = \rho\left(1 - \tau_{t+1}^k\right)R_{t+1}u'(x_{t+1} - \delta v_{t+1}^o) \tag{46}$$

and

$$n\rho u'(x_t - \delta v_t^o) \geq \left(1 - \tau_{t+1}^b\right)\beta u'(c_t - \gamma v_t^y), \tag{47}$$

for $t = 0, 1, 2, \ldots$, where the last condition holds with equality if $b_t > 0$.

The competitive equilibrium of this economy with taxes is a path $\{k_t, c_t, x_t, b_t, \phi_t^y, \phi_t^o\}_{t=0}^{\infty}$ that, for a given initial value of k_0, solves the system of difference equations composed of (46) and (47), together with (13), (14), (15), (41), (42), (43), (44), (45) and the transversality condition (21) for a given path of $\{\tau_t^y, \tau_t^o, \tau_t^b, \tau_t^k\}_{t=0}^{\infty}$.

Let us denote the path of optimal tax rates as $\left\{\tilde{\tau}_t^y, \tilde{\tau}_t^o, \tilde{\tau}_t^b, \tilde{\tau}_t^k\right\}_{t=0}^{\infty}$. These optimal tax rates are such that make the equilibrium path of $\{k_t, c_t, x_t\}$ coincide with the optimal path $\{k_t^p, c_t^p, x_t^p\}$, characterized by (29), (33) and (34). By using (3), we obtain that the optimal path of bequest implied by the planners' solution is given by

$$b_t^p = f'(k_t^p)k_t^p - \frac{x_t^p}{n}. \tag{48}$$

A positive optimal amount of bequest implies that the bequest motive is operative along an equilibrium path that attains the first best solution and a negative optimal amount of bequest implies that the bequest motive is inoperative along this equilibrium path. To characterize the optimal tax rates, we must distinguish between these two cases.

When the optimal amount of bequest is positive, the equilibrium path associated to the optimal tax rates is characterized by (29), (45) and the following two equations, which are obtained from combining (46) and (47):

$$R(k_{t+1})\left(\frac{u'(\hat{c}_{t+1})}{u'(\hat{c}_t)}\right) = \left(\frac{n}{\beta}\right)\left(\frac{1}{\left(1 - \tilde{\tau}_{t+1}^b\right)\left(1 - \tilde{\tau}_{t+1}^k\right)}\right), \tag{49}$$

and

$$\left(\frac{n\rho}{\beta}\right)\left(\frac{u'(\hat{x}_t)}{u'(\hat{c}_t)}\right) = 1 - \tilde{\tau}_t^b. \tag{50}$$

From comparing the pair of Eqs. (49) and (50) with (33) and (34), we obtain the following result:

Proposition 6.1. *When the optimal amount of bequest is positive, the optimal tax rates are $\tilde{\tau}_t^b = 1 - I$ and $\tilde{\tau}_t^k = 1 - \frac{1}{I}$.*

If the optimal amount of bequest is positive and consumption externalities are the only source of suboptimality, then lump-sum taxes are irrelevant due to Ricardian equivalence. In this case, because the specification of the consumption externality allows a different level of distortion in youth and old age, an appropriate tax on bequests (estate tax) could outweigh the effect of externalities (see (50)). However, the estate tax affects the degree of capital accumulation according to Eq. (49). Therefore, if the optimal tax rate on bequests is positive, we should subsidize capital accumulation so as not to create a distortion where none originally existed. Conversely, if a subsidy on bequests counterbalances the effect of consumption externalities, we should introduce positive taxation on capital income to bring the amount of capital to its optimal level. In this respect note that the sign of $\tilde{\tau}_t^k$ is the opposite of that of $\tilde{\tau}_t^b$.

When the optimal amount of bequest is negative, the equilibrium path associated with the optimal tax rates is characterized by $b_t = 0$ and Eqs. (29), (45) and (46). We proceed to characterize the optimal tax rates. From the comparison between (40) and (46), it follows that the optimal capital income tax rate is $\tilde{\tau}_t^k = 1 - \frac{1}{I}$. This tax solves the suboptimality due to consumption externalities, whereas the suboptimality due to the inoperativeness of the bequest motive is thus solved through lump-sum taxes. To obtain the latter optimal taxes, we use (13), (15), (43) and (45) to obtain

$$x_t = nf'(k_t)k_t - \tau_t^o. \tag{51}$$

From the comparison between (48) and (51), we obtain that $\tilde{\tau}_t^o = nb_t^p$ and, using (41), we get $\tilde{\tau}_t^y = -b_t^p$, where b_t^p is defined in Eq. (48). Given that $b_t^p < 0$, we obtain that $\tilde{\tau}_t^o < 0$ and $\tilde{\tau}_t^y > 0$. Thus, the optimal system of lump-sum taxes corrects the excess of capital by means of reducing the income of the young agents and increasing the income of the old agents. In this sense, the optimal lump-sum taxes are a pay-as-you-go social security system. These results are summarized in the following proposition:

Proposition 6.2. *When the optimal amount of bequest is negative, the optimal tax rates are $\tilde{\tau}_t^k = 1 - \frac{1}{I}$, $\tilde{\tau}_t^o = nb_t^p$ and $\tilde{\tau}_t^y = -b_t^p$.*

We have thus shown that if agents are altruistic the optimal tax policy depends on the operativeness of the bequest motive along an optimal path. On the one hand, if bequests are zero then, as in Abel (2005), the optimal tax policy consists of a tax on capital income and a pay-as-you go social security system. The latter solves the excess of capital accumulation due to the inoperativeness of the bequest motive and the former solves the suboptimal allocation of consumption due to consumption externalities. Note that the optimal capital income tax rate will be positive or negative, depending on the relative intensity of the two externalities. Thus, this optimal tax is constructed in a way that stimulates or deters saving depending on the overall effect of externalities on the allocation of consumption.

On the other hand, when the bequest motive is operative, the only source of deviation from optimality comes from the presence of consumption spillovers. In this case, lump-sum taxes are irrelevant and the optimal tax policy consists of an estate tax and a capital income tax. Thus, when generations are effectively linked by means of bequests, both a capital income tax and an estate tax must be introduced in order to improve the suboptimal allocation arising from consumption externalities. The former induces the optimality of the amount of saving and the latter brings bequests at their optimal level. The two tax rates have opposite signs.

Finally, note that the optimal level of the tax rate on capital income is the same in the two scenarios we have just considered even though it is aimed at fixing a deviation from optimality coming through different channels.

7. Conclusion

In this paper we have analyzed the suboptimality due to the existence of consumption externalities in a dynastic altruism model. We have shown that the operativeness of the bequest motive changes the nature of the suboptimality due to consumption externalities. When bequests are positive, consumption spillovers only give raise to an intratemporal deviation from optimality, whereas they are a source of intertemporal suboptimality when bequests are zero. In the former case, consumption externalities only affect the intratemporal margin between consumption of the two generations living in the same period. This implies that consumption externalities only affect the optimality of the allocation of consumption between the two living generations, which is associated with a suboptimal level of bequest. Thus, the intertemporal paths of aggregate consumption, saving and output remain at their optimal levels. In contrast, when bequests are zero, consumption externalities affect the intertemporal margin between consumption when young and when old. Given that consumption externalities affect the intertemporal path of aggregate consumption, they also modify the path of capital and production. In fact, the externality associated with the young consumption reference reduces the stock of capital and thus reduces the excess of capital accumulation. Obviously, the externality associated with the old consumption reference has the opposite effect on the stock of capital.

We have characterized the optimal tax rates and we have shown that they depend on the operativeness of the bequest motive. On the one hand, when the bequest motive is inoperative, the optimal tax policy consists of a tax on capital income and a pay-as-you go social security system. On the other hand, when the bequest motive is operative, the optimal tax policy consists of an estate tax and a capital income tax. Therefore, consumption externalities justify the introduction of estate taxes.

We have shown that the optimal capital income and estate tax rates are constant along the transition. This is simply a consequence of the fact that externalities accrue from contemporaneous consumption. If we had assumed instead that the reference levels of consumption are related to the past average consumption in the economy (external habits or aspirations), then the optimal tax rates would not be constant along the transition. This seems a promising line of future research as the absolute separation between intratemporal and intertemporal suboptimality will not hold when externalities accruing from past consumption are introduced. Finally, let us mention that other tax instruments can also solve the suboptimality due to consumption externalities. In particular, taxes on consumption that discriminate between generations may also implement an optimal allocation. The characterization of optimal consumption taxes is thus another interesting extension even though the results would be critically dependent on the functional forms of the utility and production functions.

Appendix A. Proofs

Proof of Proposition 3.3. The amount of bequests in a steady state equilibrium b^* is strictly positive only if (20) holds with equality. In this case, the steady state is characterized by this equation and $h(k, b)=0$. From these two equations (or, equivalently, from (19) and (20)), it follows that the corresponding steady state capital stock k^* satisfies

$$R(k^*) = \frac{n}{\beta}. \tag{52}$$

The uniqueness of the value k^* follows immediately from the properties imposed on the production function. Recall that, by assumption, $h(k, 0) \gtrless 0$ whenever $k \gtrless \bar{k}$ and, from Lemma 3.1, $h(k, b)$ is decreasing in b. Therefore, $h(k, b^*) < h(k, 0)$ for all $k > 0$ and $b^* > 0$. Therefore, the value k^* satisfying $h(k^*, b^*)=0$ must be strictly larger than $\bar{k}$ or, equivalently $R(k^*) < R(\bar{k})$. From (52), we see that this means that $R(\bar{k}) > \frac{n}{\beta}$, which in turn is equivalent to $\beta > \bar{\beta}$ (see (26)). We have thus proved that $\beta > \bar{\beta}$ if and only if $b^* > 0$ and $f'(k^*) = \frac{n}{\beta}$. Therefore, $\beta \leq \bar{\beta}$ implies that the stationary amount of bequests is zero and, hence, the steady state capital stock $\bar{k}$ satisfies $h(\bar{k}, 0)=0$. This value $\bar{k} > 0$ exists and is unique by assumption.[7]   □

Proof of Proposition 3.4.

a) Stability when the bequest motive is not operative

We first derive conditions aimed at guaranteeing the local stability of the steady state with zero bequests. Note that in a neighborhood of this steady state the following inequality is satisfied:

$$n\rho u'(x_t - \delta v_t^o) > \beta u'(c_t - \gamma v_t^y),$$

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[7] See Thibault (2000) for a discussion on the existence of multiple steady states in a model without consumption externalities.

so that $b_t = 0$ for $t = 0, 1, 2, \ldots$. Then, the transitional dynamics around the steady state is characterized by (19), which can be rewritten as

$$h_t \equiv u'(c_t - \gamma v_t^y) - \rho R_{t+1} u'(x_{t+1} - \delta v_{t+1}^o) = 0, \quad \text{for} \quad t = 0, 1, 2, \ldots.$$

By using (2), (3), (9), (10) and (13)–(15), it is immediate to see that we can obtain the relationship $h_t = \hat{h}(k_t, k_{t+1}, k_{t+2})$. Next, we obtain the following derivatives evaluated at the steady state:

$$\frac{\partial h_t}{\partial k_t} = u'(\hat{c}) \left(-\sigma(\hat{c}) \frac{\frac{\partial \hat{c}}{\partial k_t}}{\hat{c}} + \pi(\hat{x}) \frac{\frac{\partial \hat{x}_{t+1}}{\partial k_t}}{\hat{x}} \right),$$

$$\frac{\partial h_t}{\partial k_{t+1}} = u'(\hat{c}) \left(-\sigma(\hat{c}) \frac{\frac{\partial \hat{c}}{\partial k_{t+1}}}{\hat{c}} + \pi(\hat{x}) \frac{\frac{\partial \hat{x}_{t+1}}{\partial k_{t+1}}}{\hat{x}} + \alpha(k) \right),$$

$$\frac{\partial h_t}{\partial k_{t+2}} = \pi(\hat{x}) u'(\hat{c}) \left(\frac{\frac{\partial \hat{x}_{t+1}}{\partial k_{t+2}}}{\hat{x}} \right),$$

where $\sigma(\hat{c}) = -\frac{\hat{c} u''(\hat{c})}{u'(\hat{c})} > 0$, $\pi(\hat{x}) = -\frac{\hat{x} u''(\hat{x})}{u'(\hat{x})} > 0$ and $\alpha(k) = -\frac{R'(k)k}{R(k)} > 0$. The stability can be characterized by using the characteristic polynomial

$$P(\lambda) = \frac{\partial h_t}{\partial k_{t+2}} \lambda^2 + \frac{\partial h_t}{\partial k_{t+1}} \lambda + \frac{\partial h_t}{\partial k_t}.$$

Local saddle path stability implies that only one of the two characteristic roots must belong to the unit circle. Note also that $P(1) = h_k(\bar{k}, 0) > 0$. Then, saddle path stability requires that $P(-1) < 0$, where

$$P(-1) = u'(\hat{c}) \left[\left(\frac{\pi(\hat{x})}{\hat{x}} \right) \left(\frac{\partial \hat{x}_{t+1}}{\partial k_{t+2}} + \frac{\partial \hat{x}_{t+1}}{\partial k_t} - \frac{\partial \hat{x}_{t+1}}{\partial k_{t+1}} \right) + \left(\frac{\sigma(\hat{c})}{\hat{c}} \right) \left(\frac{\partial \hat{c}}{\partial k_{t+1}} - \frac{\partial \hat{c}}{\partial k_t} \right) - \alpha(k) \right].$$

By using (2), (3), (5) and (6), $P(-1)$ can be rewritten as

$$P(-1) = u'(\hat{c}) \left[\left(\frac{\pi(\hat{x})}{\hat{x}} \right) [\delta \varepsilon^o (n - R'(k)k) - (1 - \delta(1 - \varepsilon^o)) n (R(k) + R'(k)k)] + \left(\frac{\sigma(\hat{c})}{\hat{c}} \right) [\gamma(1 - \varepsilon^y) n (R(k) + R'(k)k) - (1 - \gamma \varepsilon^y)(n - R'(k)k)] - \alpha(k) \right].$$

Condition (25) implies that $P(-1) < 0$ for values of δ and γ sufficiently close to zero.

b) Stability when the bequest motive is operative

Note that (20) implicitly defines $\hat{x}_t = \phi(\hat{c}_t)$, where the derivative of this implicit function is

$$\phi' = \left(\frac{u'(\hat{x}_t)}{u''(\hat{x}_t)} \right) \left(\frac{u''(\hat{c}_t)}{u'(\hat{c}_t)} \right) > 0.$$

Next, combine

$$c_t = (1 - \gamma \varepsilon^y) c_t - \gamma(1 - \varepsilon^y) x_t,$$

and

$$\hat{x}_t = (1 - \delta(1 - \varepsilon^o)) x_t - \delta \varepsilon^o c_t,$$

to obtain

$$c_t = \frac{[1 - \delta(1 - \varepsilon^o)] \hat{c}_t + \gamma(1 - \varepsilon^y) x_t}{(1 - \gamma \varepsilon^y)[1 - \delta(1 - \varepsilon^o)] - \gamma \delta \varepsilon^o (1 - \varepsilon^y)},$$

and

$$x_t = \frac{(1 - \gamma \varepsilon^y) \hat{x}_t + \delta \varepsilon^o \hat{c}_t}{(1 - \gamma \varepsilon^y)[1 - \delta(1 - \varepsilon^o)] - \gamma \delta \varepsilon^o (1 - \varepsilon^y)}.$$

Using the previous equations, the resource constraint

$$f(k_t) = c_t + \frac{x_t}{n} + n k_{t+1}$$

can be rewritten as

$$k_{t+1} = \frac{f(k_t)}{n} - \left(\frac{[n(1 - \delta(1 - \varepsilon^o)) + \delta \varepsilon^o] \hat{c}_t + [1 - \gamma \varepsilon^y + n \gamma(1 - \varepsilon^y)] \phi(\hat{c}_t)}{(1 - \gamma \varepsilon^y)[1 - \delta(1 - \varepsilon^o)] - \gamma \delta \varepsilon^o (1 - \varepsilon^y)} \right) \left(\frac{1}{n^2} \right). \tag{53}$$

Finally, combining (19) and (20) we obtain

$$u'(\hat{c}_{t+1}) = \frac{nu'(\hat{c}_t)}{\beta R(k_{t+1})},$$

which implicitly defines the relationship

$$\hat{c}_{t+1} = \kappa(\hat{c}_t, k_{t+1}). \tag{54}$$

Note that (53) and (54) form the system of difference equations that completely drives the local transition of the equilibrium path. The elements of the Jacobian matrix of this system are

$$\frac{\partial k_{t+1}}{\partial k_t} = \frac{f'(k)}{n} = \frac{1}{\beta} > 0,$$

$$\frac{\partial k_{t+1}}{\partial \hat{c}_t} = -\left(\frac{[n(1-\delta(1-\varepsilon^o)) + \delta\varepsilon^o] + [1-\gamma\varepsilon^y + n\gamma(1-\varepsilon^y)]\phi'(\hat{c}_t)}{(1-\gamma\varepsilon^y)[1-\delta(1-\varepsilon^o)] - \gamma\delta\varepsilon^o(1-\varepsilon^y)}\right)\left(\frac{1}{n^2}\right) < 0,$$

$$\frac{\partial \hat{c}_{t+1}}{\partial k_t} = -\left(\frac{nu'(\hat{c})R'(k)}{u''(\hat{c})\beta(R(k))^2}\right)\left(\frac{\partial k_{t+1}}{\partial k_t}\right) = -\left(\frac{u'(\hat{c})}{u''(\hat{c})}\right)\left(\frac{f''(k)}{f'(k)}\right)\left(\frac{\partial k_{t+1}}{\partial k_t}\right) < 0,$$

and

$$\frac{\partial \hat{c}_{t+1}}{\partial \hat{c}_t} = \frac{nu''(\hat{c})}{u''(\hat{c})\beta R(k)} - \left(\frac{nu'(\hat{c})R'(k)}{u''(\hat{c})\beta(R(k))^2}\right)\left(\frac{\partial k_{t+1}}{\partial c_t}\right) = 1 - \left(\frac{u'(\hat{c})}{u''(\hat{c})}\right)\left(\frac{f''(k)}{f'(k)}\right)\left(\frac{\partial k_{t+1}}{\partial \hat{c}_t}\right).$$

Define the characteristic polynomial as

$$Q(\lambda) = \lambda^2 - \left(\frac{1}{\beta} + 1 - \left(\frac{u'(\hat{c})}{u''(\hat{c})}\right)\left(\frac{f''(k)}{f'(k)}\right)\left(\frac{\partial k_{t+1}}{\partial \hat{c}_t}\right)\right)\lambda + \frac{1}{\beta}.$$

Note that

$$Q(1) = \left(\frac{u'(\hat{c})}{u''(\hat{c})}\right)\left(\frac{f''(k)}{f'(k)}\right)\left(\frac{\partial k_{t+1}}{\partial \hat{c}_t}\right) < 0,$$

and

$$Q(-1) = 2\left(1 + \frac{1}{\beta}\right) - \left(\frac{u'(\hat{c})}{u''(\hat{c})}\right)\left(\frac{f''(k)}{f'(k)}\right)\left(\frac{\partial k_{t+1}}{\partial \hat{c}_t}\right) > 0.$$

This implies that there is a unique root within the unit circle, which proves the desired local saddle path stability. □

Proof of Proposition 3.5. a) Since bequests are positive in this case, the capital stock k^* is such that $R(k^*) = \frac{n}{\beta}$ and b^* satisfies $h(k^*, b^*) = 0$. It is obvious that $\frac{\partial k^*}{\partial \gamma} = \frac{\partial k^*}{\partial \delta} = 0$. To see the effect on b^*, we use $h(k^*, b^*) = 0$. On the one hand, we use Lemma 3.1 to obtain the following: $\frac{\partial b^*}{\partial \gamma} = -\frac{h_\gamma}{h_b} > 0$ and $\frac{\partial b^*}{\partial \delta} = -\frac{h_\delta}{h_b} < 0$, where

$$h_\gamma \equiv \frac{\partial h}{\partial \gamma} = -u''(\hat{c})\underbrace{\left[\varepsilon^y\left(w\left(k^*\right) + b^* - nk^*\right) + (1-\varepsilon^y)n\left(R\left(k^*\right)k^* - b^*\right)\right]}_{+} < 0$$

and

$$h_\delta \equiv \frac{\partial h}{\partial \delta} = \rho R\left(k^*\right)u''(\hat{x})\underbrace{\left[(1-\varepsilon^o)n\left(R\left(k^*\right)k^* - b^*\right) + \varepsilon^o\left(w\left(k^*\right) + b^* - nk^*\right)\right]}_{+} < 0.$$

b) In this case the amount of bequest is zero. Then, the steady state capital stock solves $h(\bar{k}, 0) = 0$. Using this equation, we obtain that $\frac{\partial \bar{k}}{\partial \gamma} = -\frac{h_\gamma}{h_k}$ and $\frac{\partial \bar{k}}{\partial \delta} = -\frac{h_\delta}{h_k}$, where $h_k\left(\bar{k}, 0\right) > 0$ from Assumption A,

$$h_\gamma = -u''(\hat{c})\underbrace{\left[\varepsilon^y\left(w\left(\bar{k}\right) - n\bar{k}\right) + (1-\varepsilon^y)Rn\bar{k}\right]}_{+} > 0$$

and

$$h_\delta = \rho R\left(\bar{k}\right)u''(x)\underbrace{\left[(1-\varepsilon^o)R\left(\bar{k}\right)nk + \varepsilon^o\left(w\left(\bar{k}\right) - n\bar{k}\right)\right]}_{+} < 0.$$

The results then follow immediately. □

Proof of Proposition 3.6. Obvious from Eq. (26) and Proposition 3.5. □

Proof of Proposition 4.1. The proof follows from (29), (33), (34) and by noticing that (36) defines an increasing relation between x^p and c^p and (37) defines a decreasing relation between these two variables. □

Appendix B. Numerical example

Let us assume an instantaneous logarithmic utility function,

$$u(z_t) = \ln(z_t),$$

and a Cobb–Douglas production function

$$f(k) = Ak^\mu.$$

Concerning the value of the parameters, we assume that $\rho=0.045$, $\mu=0.35$, $n=m^{35}$, with $m=1.01$, and $A=1$. We also give equal weight to the two living agents, i.e., $\theta^o = \theta^y$.

Table 1
This table shows the combinations of parameter values for which the equilibrium exhibits saddle path stability

	$\theta^y = \theta^o = 0$	$\theta^y = \theta^o = 0.5$	$\theta^y = \theta^o = 1$
$\delta=0$	$\gamma \in [0,1]$	$\gamma \in [0,0.993)$	$\gamma \in [0,0.989)$
$\delta=0.5$	$\gamma \in [0,1]$	$\gamma \in [0,0.992)$	$\gamma \in [0,0.986)$
$\delta=0.75$	$\gamma \in [0,1]$	$\gamma \in [0,0.987)$	$\gamma \in [0,0.965)$
$\delta=0.85$	$\gamma \in [0,1]$	$\gamma \in [0,0.943)$	$\gamma \in [0,0.575)$
$\delta=1$	$\gamma \in [0,0.999)$	$\emptyset$	$\emptyset$

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