

# Extensions of the Shapley value for Environments with Externalities\*

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October 2017

## Abstract

Shapley (1953a) formulates his proposal of a value for cooperative games with transferable utility in characteristic function form, that is, for games where the resources every group of players has available to distribute among its members only depend on the members of the group. However, the worth of a coalition of agents often depends on the organization of the rest of the players. The existence of externalities is one of the key ingredients in most interesting economic, social, or political environments. Thrall and Lucas (1963) provide the first formal description of settings with externalities by introducing the games in partition function form. In this chapter, we present the extensions of the Shapley value to this larger set of games. The different approaches that lead to the Shapley value in characteristic function form games (axiomatic, marginalistic, potential, dividends, non-cooperative) provide alternative routes for addressing the question of the most suitable extension of the Shapley value for the set of games in partition function form.

*JEL classification:* C71, D62.

*Keywords:* Shapley value, Externalities.

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\*We thank Chaoran Sun for helpful comments. We gratefully acknowledge financial support from the Ministerio de Economía y Competitividad and Feder (ECO2015-63679-P), Generalitat de Catalunya (ICREA Academia), and Severo Ochoa Programme (SEV-2015-0563).

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# 1 Introduction

As is emphasized in the other chapters of this book, the Shapley value, a central concept in cooperative game theory, addresses the question of how players should share the gains from cooperation. Shapley (1953a) formulates his proposal for cooperative games with transferable utility in characteristic function form, that is, for games where the resources every group of players has available to distribute among its members depends exclusively on the actions of the group members. His proposal has important applications in economics, such as the study of markets with given sets of potential buyers and potential sellers.<sup>1</sup>

However, describing environments through characteristic function form games may imply an important shortcoming, since the worth of a coalition of players often depends on the actions of players outside the group. In fact, the existence of such external effects is one of the key ingredients in most economic, social, or political environments. To mention just a few examples, in treaty agreements the gain of the participant countries depends on the way the non-member countries act, that is, on whether they form a union or they partition into singletons. In economic or political mergers the gain of the participants in the integration depends on the arrangements reached by the non-included firms or political parties. For cartels and research joint ventures, there are important cross effects, since what a group of players obtains depends on the groups formed by the other players.

The abundance of situations where externalities among coalitions are present calls for extending the class of cooperative games to allow for the presence of such cross effects. The first formal description of settings with externalities is provided by Thrall and Lucas (1963), who introduce games in partition function form. Since then, several cooperative solution concepts, and most notably the Shapley value, have been extended to games with externalities.

In this chapter, we present the extensions of the Shapley value to games in partition

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<sup>1</sup>In addition, as several authors have underlined, the fact that the Shapley value can be interpreted in terms of “marginal contributions” makes it perhaps the game theoretic concept most closely related to traditional economic ideas (see, e.g., Aumann, 1994).

function form. One possible avenue to address the task of extending the value is to take the Shapley value axioms for games in characteristic function form and adapt them to that larger class of games. The extension of the Shapley value axioms has to take a stand on the treatment (importance) of the various externalities. Different approaches to these issues lead to distinct systems of axioms, in particular distinct dummy player axioms, all of which reduce to the original Shapley axioms in the absence of externalities. As a consequence, several plausible extensions of the Shapley value are obtained.

Myerson (1977) is the first attempt to extend the Shapley value for games in partition function form. As we will see later, his set of axioms identifies a unique value. However, in environments where externalities are present, natural extensions of the Shapley axioms do not necessarily imply a unique value. That is why most authors have imposed additional and/or different axioms to identify a unique solution (Bolger, 1989; Albizuri, Arin, Rubio, 2005; Macho-Stadler, Pérez-Castrillo, Wettstein, 2007; Pham Do and Norde, 2007; McQuillin, 2009; Hu and Yang, 2010; Grabisch and Funaki, 2012).

Other possible avenues to extend the Shapley value to games in partition function form are based on alternative ways to characterize the Shapley value, such as the marginalistic approach (De Clippel and Serrano, 2008), the potential avenue (Dutta, Ehlers and Kar, 2010) and the algorithmic route.

The chapter is organized as follows. In section 2 we present the environment, and in section 3 the proposals for extending the Shapley value for games with externalities using the axiomatic approach. Section 4 presents the extensions of the value based on the agents' marginal contributions. Section 5 describes extensions that follow the approaches of the potential, the Harsanyi dividends, and the algorithmic view. Section 6 provides non-cooperative foundations to several values for partition function form games. A concluding section offers some examples of applications and avenues for future research.

## 2 The environment

Cooperative games with externalities were first introduced by Thrall and Lucas (1963) as transferable utility (TU)  $n$ -person games in *partition function form* (PFF) as follows.

Given a set of players,  $N = \{1, \dots, n\}$ , a coalition  $S$  is a group of  $s$  players, that is, a non-empty subset of  $N$ ,  $S \subseteq N$ . An embedded coalition specifies the coalition,  $S$ , as well as the structure of coalitions formed by the other players, that is, an embedded coalition is a pair  $(S, P)$ , where  $S$  is a coalition and  $P \ni S$  is a partition of  $N$ . We adopt the convention that the empty set  $\emptyset$  is in  $P$  for every partition  $P$  although we refrain from explicitly inserting it in the partitions. A particular partition is  $[N] = \{\{i\}_{i \in N}\}$ , where all the coalitions are singleton coalitions. More generally, we denote by  $[S]$  the partition of  $S$  consisting of all the singleton players in  $S$ , that is,  $[S] = \{\{i\}_{i \in S}\}$ .

Let  $\mathcal{P}(N)$  denote the set of all partitions of  $N$  and  $\mathcal{P}_S = \{P \in \mathcal{P}(N) \mid S \in P\}$  the set of partitions including  $S$ . The set of embedded coalitions of  $N$  is denoted by  $ECL$ :

$$ECL = \{(S, P) \mid P \in \mathcal{P}_S \text{ and } S \subseteq N\}.$$

A PFF game is given by a set of players,  $N$ , and a function,  $v : ECL \rightarrow \mathbb{R}$ , that associates a real number with each embedded coalition. Thus,  $v(S, P)$  is the worth of coalition  $S$  when the players are organized according to the partition  $P$ . We assume that  $v(\emptyset, P) = 0$ . Let  $\mathcal{G}^N$  be the set of games in PFF with players in  $N$ . We will sometimes refer to some particularly simple games which we will denote by  $(N, w_{(S,P)})$ . The function  $w_{(S,P)}$  is defined as  $w_{(S,P)}(S, P) = w_{(S,P)}(N, \{N\}) = 1$  and  $w_{(S,P)}(S', P') = 0$  for any  $(S', P')$  different from  $(S, P)$  and  $(N, \{N\})$ .<sup>2</sup>

Some games in  $\mathcal{G}^N$  do not have externalities. A game has no externalities if the worth of any coalition  $S$  is independent of the way the other players are organized. A game with no externalities satisfies  $v(S, P) = v(S, P')$  for any  $P, P' \in \mathcal{P}_S$  and any coalition  $S \subseteq N$ . We denote a game with no externalities by  $\hat{v}$ . Since in this case the worth of a coalition  $S$  can be written without reference to the organization of the remaining players, we can write  $\hat{v}(S) \equiv \hat{v}(S, P)$  for all  $P \in \mathcal{P}_S$  and all  $S \subseteq N$  for such games. We denote by  $G^N$  the set of games without externalities with players in  $N$ , which corresponds to the set of TU games in *characteristic function form* (CFF). For convenience, we will denote a value for games in characteristic form by  $\psi$ , that is,  $\psi : G^N \rightarrow \mathbb{R}^n$ . We denote the Shapley value for a CFF game  $\hat{v}$  by  $\psi^{Sh}(\hat{v})$ .

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<sup>2</sup>The set of games  $\{w_{(S,P)}\}_{(S,P) \in ECL}$  constitutes a basis for  $\mathcal{G}^N$ .

After the introduction of PFF games by Thrall and Lucas (1963), the subsequent literature dealt with both the structure of multi-valued solutions and the construction of single-valued solutions for PFF games. A single-valued solution is given by a function  $\varphi : \mathcal{G}^N \rightarrow \mathbb{R}^n$ , where  $\varphi_i(v)$  is the payoff assigned by the solution  $\varphi$  to player  $i \in N$  in the PFF game  $v$ . As mentioned in the Introduction, in this chapter we are interested in the extensions of the Shapley value  $\psi^{Sh}(\hat{v})$  for PFF games.

### 3 Axiomatic extensions of the Shapley value for games with externalities

One branch of the literature takes as a starting point the axioms underlying the Shapley value for CFF games. These axioms can be extended to PFF games in several ways and give rise to several distinct “Shapley-like” values. New axioms can also be proposed to deal with the externalities.

First of all, let us note that all the values we present in this section assume that the grand coalition will form and thus the value will share the worth of the grand coalition, that is, the value is efficient.<sup>3</sup>

**Efficiency axiom.** A value  $\varphi$  is efficient if  $\sum_{i \in N} \varphi_i(N, v) = v(N, \{N\})$  for any  $v \in \mathcal{G}^N$ .

Myerson (1977) was the first to extend the Shapley axioms to PFF games and obtain a value for this class of games. The symmetry and additivity axioms were extended in the following, natural way. Let us define the  $\sigma-$  permutation of the game  $v \in \mathcal{G}^N$ , denoted by  $\sigma v$ , as  $(\sigma v)(S, P) \equiv v(\sigma S, \sigma P)$  for all  $(S, P) \in ECL$ .

**Symmetry axiom.** A value  $\varphi$  is symmetric if  $\varphi(\sigma v) = \sigma \varphi(v)$  for any  $v \in \mathcal{G}^N$  and for any permutation  $\sigma$  of  $v$ .

This symmetry axiom is interpreted as an anonymity axiom.

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<sup>3</sup>This may be the most adequate assumption for games where the grand coalition maximizes joint surplus. Hafalir (2007) shows that a natural extension of superadditivity for PFF games is not sufficient to imply that the grand coalition is efficient, and provides a condition, analogous to convexity, that is sufficient for a game to have this feature.

If we define the addition of two games  $v$  and  $v'$  in  $\mathcal{G}^N$  as the game  $v + v'$ , where  $(v + v')(S, P) \equiv v(S, P) + v'(S, P)$  for all  $(S, P) \in ECL$ , then the additivity axiom can be written as follows:

**Additivity axiom.** A value  $\varphi$  is additive if  $\varphi(v+v') = \varphi(v)+\varphi(v')$  for any  $v, v' \in \mathcal{G}^N$ .

In Myerson (1977), the dummy and efficiency axioms are extended by providing a carrier definition for PFF games. We say that  $S \subseteq N$  is a *carrier* for  $v$  if and only if

$$v(\tilde{S}, \tilde{Q}) = v(\tilde{S} \cap S, \tilde{Q} \wedge \{S, N \setminus S\}) \text{ for every } (\tilde{S}, \tilde{Q}) \in ECL.$$

That is,  $S$  is a carrier for  $v$  if the payoff of any embedded coalition  $(\tilde{S}, \tilde{Q})$  is determined by the set of players in  $\tilde{S}$  that are in  $S$  and the meet  $\tilde{Q} \wedge \{S, N \setminus S\}$  of the partitions  $\tilde{Q}$  and  $\{S, N \setminus S\}$  (the largest partition that refines both). The carrier axiom for CFF games is then extended as follows:

**Carrier axiom.** A value  $\varphi$  satisfies the carrier axiom if  $\sum_{i \in S} \varphi_i(N, v) = v(N, \{N\})$  for any  $v \in \mathcal{G}^N$  for which  $S$  is a carrier.

The three axioms of symmetry, additivity, and carrier yield a unique value, allowing Myerson (1977) to propose the extension  $\varphi^M(v)$  given by

$$\varphi_i^M(v) = \sum_{(S,P) \in ECL} (-1)^{|P|-1} (|P|-1)! \left[ \frac{1}{n} - \sum_{\substack{T \in P \setminus \{S\} \\ i \notin T}} \frac{1}{(|P|-1)(n-|T|)} \right] v(S, P)$$

for any  $i \in N$ , where  $|T|$  is the number of agents in  $T$  and  $|P|$  is the number of non-empty coalitions in  $P$ .<sup>4</sup>

While the extension of the efficiency axiom through the carrier axiom is natural, the extension of the dummy player axiom may be more problematic. A player  $i \in N$  is a dummy player, in the sense of Myerson (1977), if there exists a carrier  $S$  with  $i \notin S$ . The carrier axiom implies that such a dummy player will receive zero according to  $\varphi^M$ . This is problematic since a dummy player, thus defined, might have an effect on the worth of coalitions. Take, for example, the game with three players  $(\{1, 2, 3\}, w_{(\{1\}, \{\{1\}, \{2, 3\}\})})$ . In this game, player 1 is a carrier and hence players 2 and 3 are dummy players. Therefore,

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<sup>4</sup> Albizuri (2010) adapts the axioms in Myerson (1977) to extend  $\varphi^M$  to a new class of games, where players can take part in more than one coalition, named “games in coalition configuration function form.”

$\varphi_1^M(w_{(\{1\}, \{\{1\}, \{2,3\}\})}) = 1$  and  $\varphi_2^M(w_{(\{1\}, \{\{1\}, \{2,3\}\})}) = \varphi_3^M(w_{(\{1\}, \{\{1\}, \{2,3\}\})}) = 0$ . On the other hand, in the possibly similar game  $(\{1, 2, 3\}, w_{(\{1\}, \{\{1\}, \{2\}, \{3\}\})})$ , player 1 is not a carrier and, in fact,  $\varphi_1^M(w_{(\{1\}, \{\{1\}, \{2\}, \{3\}\})}) = 0$ .

Bolger (1989) is the second author to obtain a value for PFF games by suggesting a different extension of the Shapley axioms. The efficiency and symmetry axioms are extended as above and the additivity axiom is strengthened to a natural linearity axiom, regarding both addition and multiplication by a scalar.<sup>5</sup>

Formally, given the game  $v \in \mathcal{G}^N$  and the scalar  $\lambda \in \mathbb{R}$ , the game  $\lambda v$  is defined by  $(\lambda v)(S, P) \equiv \lambda v(S, P)$  for all  $(S, P) \in ECL$ .

**Linearity axiom.** A value  $\varphi$  is linear if it is additive and  $\varphi(\lambda v) = \lambda\varphi(v)$  for any  $v \in \mathcal{G}^N$  and for any scalar  $\lambda \in \mathbb{R}$ .

Bolger (1989) also introduces a dummy player axiom, which is a natural generalization of the dummy player axiom for CFF games. We will say that player  $i$  is a *dummy* player in  $v \in \mathcal{G}^N$  if he alone receives zero for any partition of the other players and, furthermore, he has no effect on the worth of any coalition  $S$  (i.e., the worth of  $S$  in partition  $P$  is constant for all possible assignments of player  $i$  to some coalition in  $P$ ). That is, player  $i$  is a dummy player in  $v \in \mathcal{G}^N$  if for every  $(S, P) \in ECL$  with  $i \in S$  and each  $R \in P \setminus \{S\}$ ,  $v(S, P) = v(S \setminus \{i\}, P \setminus \{S, R\} \cup \{S \setminus \{i\}, R \cup \{i\}\})$ .<sup>6</sup>

**Dummy player axiom.** A value  $\varphi$  satisfies the dummy player axiom if  $\varphi_i(v) = 0$  for any game  $v \in \mathcal{G}^N$  and any dummy player  $i$  in the game  $v$ .

The final axiom considered by Bolger (1989) is inspired by the desired behavior of the value over simple games, where  $v(S, P)$  equals either zero or one. It states that if the sum of marginal contributions of player  $i$  to any coalition in  $v \in \mathcal{G}^N$  is the same as in  $v' \in \mathcal{G}^N$ , then player  $i$  should receive the same payoff in both games. This axiom is well-suited to

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<sup>5</sup>In Myerson's (1977) extension, there is no need to introduce the linearity axiom. As is the case for CFF games, additivity together with the carrier axiom imply linearity. However, this is not true for the definitions of dummy player used in most papers (see Macho-Stadler, Pérez-Castrillo, and Wettstein, 2007, for a formal proof).

<sup>6</sup>For  $R = \emptyset$ , we slightly abuse notation by assuming that the partition  $P \setminus \{S, \emptyset\} \cup \{S \setminus \{i\}, \emptyset \cup \{i\}\}$  also includes the empty set.

simple games but it may be less intuitive for general games in PFF.

While Bolger (1989) shows that efficiency, symmetry, linearity, dummy player,<sup>7</sup> plus the additional axiom related to the sum of marginal contributions imply that there is a unique value  $\varphi^B$ , there is no closed-form expression for  $\varphi^B$

Macho-Stadler, Pérez-Castrillo, and Wettstein (2007 and 2017) introduce a new axiom, strong symmetry, in addition to the efficiency and symmetry axioms (appearing in both Myerson, 1977, and Bolger, 1989). The strong symmetry axiom strengthens the symmetry axiom by requiring that a player's payoff should not change after permutations in the set of players in  $N \setminus S$ , for any embedded coalition structure  $(S, P)$ . To illustrate its meaning, consider the following two games with four players:  $(\{1, 2, 3, 4\}, w_{(\{1\}, \{\{1\}, \{2\}, \{3, 4\}\})})$  and  $(\{1, 2, 3, 4\}, w_{(\{1\}, \{\{1\}, \{3\}, \{2, 4\}\})})$ . Strong symmetry requires that player 2 should receive the same payoff in both games. Another way to view it is that 2 should receive the same payoff as 3 and 4 in  $w_{(\{1\}, \{\{1\}, \{2\}, \{3, 4\}\})}$ . Since the roles of players 2 and 3 (or 4) are similar (because they only generate the externality if they are organized in a particular way), this axiom can be viewed as a symmetric treatment of the externalities generated by players. Put differently, exchanging the names of the players inducing externalities does not affect the payoff of any player.

Formally, given an embedded coalition  $(S, P)$ , denote by  $\sigma_{(S, P)}P$  a new partition such that  $S \in \sigma_{(S, P)}P$ , and the other coalitions result from a permutation of the set  $N \setminus S$  applied to  $P \setminus \{S\}$ . That is, in the partition  $\sigma_{(S, P)}P$ , the players in  $N \setminus S$  are reorganized in sets whose size distribution is the same as in  $P \setminus \{S\}$ . Given the permutation  $\sigma_{(S, P)}$ , the permutation of the game  $v$  denoted by  $\sigma_{(S, P)}v$  is defined by  $(\sigma_{(S, P)}v)(S, P) = v(S, \sigma_{(S, P)}P)$ ,  $(\sigma_{(S, P)}v)(S, \sigma_{(S, P)}P) = v(S, P)$ , and  $(\sigma_{(S, P)}v)(R, Q) = v(R, Q)$  for all  $(R, Q) \in ECL \setminus \{(S, P), (S, \sigma_{(S, P)}P)\}$ .

**Strong symmetry axiom.** A value  $\varphi$  satisfies the strong symmetry axiom if for any game  $v \in \mathcal{G}^N$  it is the case that

1.  $\varphi(\sigma v) = \sigma\varphi(v)$  for any permutation  $\sigma$  of  $N$ , and
2.  $\varphi(\sigma_{(S, P)}v) = \varphi(v)$  for any  $(S, P) \in ECL$  and for any permutation  $\sigma_{(S, P)}$ .

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<sup>7</sup>Sánchez-Pérez (2015) provides a representation of all the values that satisfy efficiency, symmetry, linearity, and dummy player.

Note that strong symmetry is implied by symmetry when there are just three players, but it is a more demanding property for games with more players.

The symmetry axioms above are associated with the idea of anonymity. One could instead require a different axiom, often considered in CFF games, usually called equal treatment of equals. This property requires that interchangeable players (that is, players that can be interchanged without affecting the value of any coalition) should receive the same payoff. For games in CFF, the symmetry and equal treatment axioms are equivalent for efficient and additive values.

Macho-Stadler, Pérez-Castrillo, and Wettstein (2017) introduce a strong equal treatment axiom for PFF games by defining a weak version of interchangeability: players  $i$  and  $j$  are *weakly interchangeable* in  $v \in \mathcal{G}^N$  if for all  $(S, P)$  with  $i \in S$  and  $j \in R \in P \setminus \{S\}$ ,  $v(S, P) = v((S \setminus \{i\}) \cup \{j\}, P \setminus \{S, R\} \cup \{(S \setminus \{i\}) \cup \{j\}, (R \setminus \{j\}) \cup \{i\}\})$ . That is, players  $i$  and  $j$  are weakly interchangeable in the game  $v$  if for any coalition  $S$  including one of them, switching them does not affect the value of any embedded coalition  $(S, P)$ . For example, in the game  $(\{1, 2, 3, 4\}, w_{(\{1\}, \{\{1\}, \{2\}, \{3, 4\}\})})$ , players 2, 3, and 4 are weakly interchangeable.

**Strong Equal Treatment axiom** A value  $\varphi$  satisfies the strong equal treatment axiom if  $\varphi_i(N, v) = \varphi_j(N, v)$  for any pair of weakly interchangeable players  $i$  and  $j$  in  $v$ .

Strong equal treatment and strong symmetry axioms are equivalent for linear and efficient values (Macho-Stadler, Pérez-Castrillo, and Wettstein, 2017).

An additional motivation for the strong symmetry axiom is that combined with efficiency and linearity, it provides an axiomatic foundation for the use of an intuitive approach to construct values for PFF games, namely the *average approach*, introduced in Macho-Stadler, Pérez-Castrillo, and Wettstein (2007 and 2017).<sup>8</sup> This approach assigns to each coalition an average of the surpluses it obtains in all the partitions it might belong to. In this way, it first transforms a PFF game to a CFF game. It then uses a value for CFF games to determine the payoffs of the players in the PFF game.

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<sup>8</sup>In the previous paper, the average approach was restricted to values satisfying the efficiency, linearity, and dummy player axioms whereas in the latter it was applied to all efficient and linear values.

Formally, the average approach constructs a value  $\varphi$  for PFF games using a value for CFF games  $\psi$  as follows. First, for any  $v \in \mathcal{G}^N$ , it constructs an average game  $\tilde{v} \in G^N$  by assigning to each  $S \subseteq N$  the average worth  $\tilde{v}(S) \equiv \sum_{P \in \mathcal{P}_S} \alpha(S, P)v(S, P)$ , with  $\sum_{P \in \mathcal{P}_S} \alpha(S, P) = 1$ . We refer to  $\alpha(S, P)$  as the “weight” of partition  $P$  in the computation of the value of coalition  $S \in P$ . Second, the value is defined as  $\varphi(v) = \psi(\tilde{v})$ .<sup>9</sup>

Macho-Stadler, Pérez-Castrillo, and Wettstein (2017) show that a value  $\varphi$  can be constructed through the average approach using a value for CFF games  $\psi$  that satisfies efficiency, linearity, and symmetry if and only if  $\varphi$  satisfies efficiency, linearity, and strong symmetry.<sup>10</sup> Adding the dummy player axiom to the desirable requirements for a value implies a constraint on the weights  $\alpha(S, P)$ , but still, many weighting systems are compatible with the four axioms. If we define the average game  $\tilde{v}$  using any of these weights then  $\varphi(v) = \psi^{Sh}(\tilde{v})$  is an extension of the Shapley value.

To select a single value, Macho-Stadler, Pérez-Castrillo, and Wettstein (2007) propose a similar influence axiom. This axiom guarantees that similar environments lead to similar payoffs for the players. Consider, for example, the following two games with three players:  $(\{1, 2, 3\}, w_{(\{1\}, \{\{1\}, \{2, 3\}\})})$  and  $(\{1, 2, 3\}, w_{(\{1\}, \{\{1\}, \{2\}, \{3\}\})})$ . These two games are very similar: in both only player 1 can produce some benefits alone. The only difference is that in  $w_{(\{1\}, \{\{1\}, \{2, 3\}\})}$ , players 2 and 3 need to be together for the benefits to player 1 to be realized, while in  $w_{(\{1\}, \{\{1\}, \{2\}, \{3\}\})}$ , players 2 and 3 should be separated. The similar influence axiom requires players 2 and 3 to receive the same payoff in both games.

Formally, we say that a pair of players  $\{i, j\} \subseteq N, i \neq j$ , has *similar influence* in games  $v$  and  $v'$  if  $v(T, Q) = v'(T, Q)$  for all  $(T, Q) \in ECL \setminus \{(S, P), (S, P')\}$ ,  $v(S, P) = v'(S, P')$ , and  $v(S, P') = v'(S, P)$ , where the only difference between partitions  $P$  and  $P'$  is that  $\{i\}, \{j\} \in P \setminus \{S\}$  while  $\{i, j\} \in P' \setminus \{S\}$ .

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<sup>9</sup>As will be clear, all the axiomatically-based values described in the remainder of this section satisfy this approach.

<sup>10</sup>Hence, the average approach can be used to extend both the Shapley value and several other values, such as the equal division value (van den Brink, 2007), the equal surplus value (Driessen and Funaki, 1991), the  $\lambda$ -egalitarian Shapley value (Joosten, 1996), the consensus value (Ju, Borm, and Ruys, 2007), and the family of least-square values (Ruiz, Valenciano, and Zarzuelo, 1998).

**Similar influence axiom.** A value  $\varphi$  satisfies the similar influence axiom if for any two games  $v, v' \in \mathcal{G}^N$  and for any pair of players  $\{i, j\} \subseteq N$  that has similar influence in those games, we have  $\varphi_i(v) = \varphi_i(v')$  and  $\varphi_j(v) = \varphi_j(v')$ .

The axioms of efficiency, linearity, dummy player, strong symmetry, and similar influence characterize a unique solution which can be constructed through the average approach by using the following weights:

$$\alpha^{MPW}(S, P) = \frac{\prod_{T \in P \setminus \{S\}} (|T| - 1)!}{(n - |S|)!}.$$

Note that  $\alpha^{MPW}(S, P)$  can be interpreted as the probability that partition  $P$  is formed, given that coalition  $S$  forms.<sup>11</sup> For any  $v \in \mathcal{G}^N$ , once we have computed the average game  $\tilde{v}^{MDW}$  using these weights, we obtain:

$$\varphi^{MPW}(v) = \psi^{Sh}(\tilde{v}^{MDW}).$$

This same value was proposed, but not axiomatized, by Feldman (1996). Let us finally note that  $\varphi^{MPW}$  satisfies the strong dummy axiom:

**Strong dummy player axiom.** A value  $\varphi$  satisfies the strong dummy player axiom if for any dummy player  $i$  in the game  $v$ ,  $\varphi_j(N, v) = \varphi_j(N \setminus \{i\}, v)$  for all  $j$  in  $N \setminus \{i\}$ .

The strong dummy property requires that adding or subtracting a dummy player from a game leaves the outcomes of the remaining players unchanged.<sup>12</sup>

Albizuri, Arin, and Rubio (2005) provide another extension of the Shapley value for PFF games, using the efficiency, symmetry, and additivity axioms, to which they add two additional properties. First, they introduce the oligarchy axiom (which can be viewed as a type of carrier axiom) for PFF games.

**Oligarchy axiom.** A value  $\varphi$  satisfies the oligarchy axiom if for any  $v \in \mathcal{G}^N$  for which there exists  $R \subseteq N$  such that  $v(S, P) = v(N, \{N\})$  if  $R \subseteq S$  and  $v(S, P) = 0$  if  $R \not\subseteq S$ , then  $\sum_{i \in R} \varphi_i(v) = v(N, \{N\})$ .

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<sup>11</sup>According to this interpretation, the denominator in the expression that defines  $\alpha^{MPW}(S, P)$  is the number of permutations of the players in  $N \setminus S$ . The numerator counts the number of those permutations of  $N \setminus S$  that “generate” the partition  $P$ , when we write a permutation as a cycle.

<sup>12</sup>This property is satisfied by the Shapley value in CFF games. Note also that for any efficient value, the strong dummy player axiom implies the dummy player axiom.

This axiom states that if there is a (oligarchic) coalition  $R$  in a game  $v$  such that any coalition that contains  $R$  generates the worth of the grand coalition, whereas any other embedded coalition has zero worth, then all the worth must be shared among the members of the oligarchic coalition. Thus, in some sense, this axiom implies a form of null player axiom, different from the dummy player axiom as defined above.

Finally, to introduce the last axiom, Albizuri, Arin, and Rubio (2005) consider a coalition  $S \subseteq N$  and a bijection  $\xi_S$  on  $\{(S, P) \mid P \in \mathcal{P}_S\}$ . For each  $v \in \mathcal{G}^N$ , denote by  $\xi_S v$  the game in  $\mathcal{G}^N$  such that  $(\xi_S v)(S, P) = v(\xi_S(S, P))$  for any  $P \in \mathcal{P}_S$  and  $(\xi_S v)(T, P) = v(T, P)$  for any  $T \in N \setminus S$  and any  $P \in \mathcal{P}_T$ .

**Embedded coalition anonymity axiom.** A value  $\varphi$  satisfies the embedded coalition anonymity axiom if for any bijection  $\xi_S$  on  $\{(S, P) \mid P \in \mathcal{P}_S\}$ , and for any  $v \in \mathcal{G}^N$ , it is the case that  $\varphi(\xi_S v) = \varphi(v)$ .

The embedded coalition anonymity axiom states that the determinant of the players' payoffs is the worth of the embedded coalitions, irrespective of the partitions that generate the worth.

Albizuri, Arin, and Rubio (2005) show that the axioms of efficiency, symmetry, additivity, oligarchy, and embedded coalition anonymity characterize a unique solution. It is given by the Shapley value of the CFF game derived from the PFF game by assigning to each coalition the arithmetic average of its worth for all the possible partitions it may belong to. That is, defining the game  $\tilde{v}^{AAR} \in \mathcal{G}^N$  as  $\tilde{v}^{AAR}(S) = \sum_{Q \in \mathcal{P}_S} \frac{1}{|\mathcal{P}_S|} v(S, Q)$ , the value is:

$$\varphi^{AAR}(v) = \psi^{Sh}(\tilde{v}^{AAR}).$$

In another axiomatic proposal, Pham Do and Norde (2007) use the efficiency, additivity, and strong equal treatment axioms. In addition, they introduce an extension of the dummy player axiom that is stronger than the one we previously defined, as they propose a weaker definition of a null player. They call player  $i \in N$  a null player if player  $i$ 's worth as a singleton is zero for any partition in  $\mathcal{P}_{\{i\}}$  and his marginal contribution to any other coalition is zero when he joins the coalition from being a singleton. Formally, player  $i \in N$  is a *null player* in  $v \in \mathcal{G}^N$  if  $v(\{i\}, P) = 0$  for every  $(\{i\}, P) \in ECL$  and

$$v(S \cup \{i\}, P \setminus \{S, \{i\}\} \cup \{S \cup \{i\}\}) = v(S, P) \text{ for each } (S, P) \text{ with } S \neq \{i\} \text{ and } \{i\} \in P.$$

Note that a so-defined null player can affect the worth of coalition  $S$  when he moves among coalitions other than  $S$ . Therefore, a player can be a null player but not a dummy player.

**Null player axiom.** A value  $\varphi$  satisfies the null player axiom if  $\varphi_i(v) = 0$  for any  $v \in \mathcal{G}^N$  and any null player  $i$  in  $v$ .

Pham Do and Norde (2007) show that there is a unique solution satisfying efficiency, additivity, symmetry, and null player. It is given by the Shapley value of the CFF game defined by  $\hat{v}^{PN}(S) \equiv v(S, [P \setminus \{S\}] \cup \{S\})$ :

$$\varphi^{PN}(v) = \psi^{Sh}(\hat{v}^{PN}).$$

Note that the value  $\varphi^{PN}$  ignores most of the information provided by the whole PFF game.

A simultaneous extension of both the Shapley value and the Owen value (Owen, 1977) for CFF games with an *a priori* coalition structure is provided by McQuillin (2009). He introduces the idea of an extended generalized value (EGV), which is a mapping  $\chi : \mathcal{G}^N \rightarrow \mathcal{G}^N$ . For  $v \in \mathcal{G}^N$ ,  $\chi(v)(S, P)$  is the value of coalition  $S$  in game  $v$  with an initial coalition structure given by  $P$ . When  $P = [N]$ , the corresponding function  $\chi(v)(\{i\}, [N])$  constitutes a standard value extension to PFF games. For partitions different to  $[N]$ , the values obtained extend values for CFF games with an initial coalition structure.

To obtain an EGV, McQuillin (2009) uses efficiency, symmetry, linearity, and dummy player (which he constructs via an appropriately extension of the carrier axiom). In addition, he introduces a weak monotonicity condition. Let  $w_{(S,P)}^o$  denote the function given by  $w_{(S,P)}^o(S, P) = 1$  and  $w_{(S,P)}^o(R, Q) = 0$  when  $(R, Q) \neq (S, P)$ ; then:

**Weak monotonicity axiom.** A value  $\chi$  satisfies weak monotonicity if  $\chi(w_{(S,P)}^o)(\{i\}, [N]) \geq 0$  for any  $i \in S$  and any game  $w_{(S,P)}^o$ .

Three further axioms are related to the behavior of the value in the presence of an *a priori* coalition structure. The first is the *rule of generalization*, implying that given an *a priori* coalition structure, each member of the partition is viewed as a single player. The second is the *cohesion axiom*, which requires that the payoff to any embedded coalition

$(S, P)$  depends only on the payoffs to those embedded coalitions with partitions that are coarser than  $(S, P)$ . The third strengthens the dummy axiom, through a *generalized null player axiom*, by requiring that a dummy player in  $v$  is also a dummy player in  $\chi(v)$ . The final axiom is the *recursion axiom* stating that  $\chi(\chi(v)) = \chi(v)$ ; that is, the solution is the right way to assign payoffs: once payoffs are assigned according to the solution, the solution will “agree” that these are the appropriate payoffs.

This set of axioms leads to a unique value called the extended Shapley value. It is given by the Shapley value of each player in the CFF game derived from the PFF game by  $\hat{v}^{MQ}(S) \equiv v(S, \{N \setminus S, S\})$ :

$$\varphi_i^{MQ}(v) = \chi(v)(\{i\}, [N]) = \psi_i^{Sh}(\hat{v}^{MQ}) \text{ for all } i \in N.$$

The value  $\varphi^{MQ}$  again abstracts from most of the information provided by the whole PFF game; it only takes into account the worth of a coalition  $S$  when other players form the complementary coalition  $N \setminus S$ . McQuillin (2009) interprets it in two ways. From a normative point of view, most information should indeed be ignored based on the properties the extension should satisfy. From a positive point of view, it implies an impossibility result: if all the information in the PFF game is taken into account, it is impossible to satisfy the axioms and the recursion property.

Finally, Hu and Yang (2010) extend the Shapley value using efficiency, symmetry, additivity, and introducing a demanding extension of the dummy player axiom. In their proposal, a player  $i \in N$  is an “average dummy player” if his average contribution to every coalition is zero, where the average is taken over all the possible partitions including the coalition. Then, Hu and Yang (2010) require the value to satisfy the axiom that the average dummy players must obtain zero. They show that this set of axioms characterize a unique extension of Shapley for PFF games, which can be written as follows:

$$\varphi_i^{HY}(v) = \sum_{\substack{P \in \mathcal{P}(N) \\ S \subseteq N, S \ni i}} \frac{(|S| - 1)! (n - |S|)!}{n! |\mathcal{P}(N)|} (v(S, P_{-S}) - v(S \setminus \{i\}, P_{-(S \setminus \{i\})})) ,$$

where, for  $P \in \mathcal{P}(N)$ , we denote  $P_{-S} = \{T \setminus S \mid T \in P\} \cup \{S\}$ , and similarly for  $P_{-(S \setminus \{i\})}$ .

## 4 Marginal contributions

The *marginal contribution* of a player to a coalition in a CFF game is the difference between the value of this coalition with and without the player. It can also be understood as a loss incurred by the remaining agents when the player leaves the coalition. For CFF games, the concept of marginal contribution of players plays an important role in the analysis of values both axiomatically and operationally (when calculating the values). In particular, Young (1985) proposes substituting the additivity and dummy player axioms in the characterization of the Shapley value for CFF games by a marginality axiom requiring a player's payoff to depend only on his own productivity measured by marginal contributions. He proves that the Shapley value can be formulated as the average of players' marginal contributions to all coalitions. In other words, the axioms of marginality, efficiency, and symmetry provide a characterization of the Shapley value.

The concept of marginal contribution is easily defined and computed for CFF games. However, defining marginal contributions is not straightforward for games with externalities because the change of worth of a coalition caused by an agent leaving this coalition depends on the partition in which it is embedded and on the identity of the coalition the agent joins.

De Clippel and Serrano (2008) thoroughly analyze the use of marginal contributions to determine possible sharings of the surplus generated in PFF games. Once they adopt the efficiency and symmetry axioms as above, they focus on properties related to marginal contributions. First, they consider the case where a player may join any other coalition after leaving a coalition  $S$ . When player  $i$  leaves coalition  $S$  in partition  $P$  to join another coalition  $T$  in  $P$ , the total effect on coalition  $S$  is:

$$v(S, P) - v(S \setminus \{i\}, P \setminus \{S, T\} \cup \{S \setminus \{i\}, T \cup \{i\}\}).$$

Therefore, a natural extension of Young's (1985) axiom is:

**Weak marginality axiom.** A value  $\varphi$  satisfies the weak marginality axiom if for any

two games  $v, v' \in \mathcal{G}^N$  for which

$$v(S, P) - v(S \setminus \{i\}, P \setminus \{S, T\} \cup \{S \setminus \{i\}, T \cup \{i\}\}) = \\ v'(S, P) - v'(S \setminus \{i\}, P \setminus \{S, T\} \cup \{S \setminus \{i\}, T \cup \{i\}\}),$$

for any  $(S, P) \in ECL$  with  $i \in S, T \neq S$  and  $T \in P$ , then it is the case that  $\varphi_i(v) = \varphi_i(v')$ .

The three axioms of efficiency, symmetry, and weak marginality impose very few restrictions on values satisfying them. It is possible to strengthen the weak marginality axiom to a “monotonicity axiom” which states that if a player’s marginal contributions in game  $v$  are greater than or equal to (with at least one strict inequality) the corresponding marginal contributions in game  $v'$ , then the player’s payoff in  $v$  must be greater than the payoff in  $v'$ . This new axiom, together with efficiency and symmetry, imposes upper and lower bounds on the payoffs prescribed by values satisfying them. Still, there is a large family of values satisfying the three axioms.

One way to single out a unique value is by strengthening the weak marginality axiom. To do this, De Clippel and Serrano (2008) decompose the total effect on coalition  $S$  when player  $i$  leaves  $S$  in  $P$  to join another  $T \in P$  in the “intrinsic marginal contribution,” given by  $v(S, P) - v(S \setminus \{i\}, P \setminus \{S\} \cup \{S \setminus \{i\}, \{i\}\})$ , and the “externality effect,” given by  $v(S \setminus \{i\}, P \setminus \{S\} \cup \{S \setminus \{i\}, \{i\}\}) - v(S \setminus \{i\}, P \setminus \{S, T\} \cup \{S \setminus \{i\}, T \cup \{i\}\})$ . That is, the intrinsic marginal contribution is the loss incurred due to the player leaving  $S$  and becoming a singleton. The externality effect is the additional loss incurred when the player joins coalition  $T$ .

Then, De Clippel and Serrano (2008) introduce a “marginality axiom” stating that the value assigned to player  $i$  depends only on the intrinsic marginal contributions of the player. The value characterized by the marginality axiom together with efficiency and symmetry coincides with the one proposed by Pham Do and Norde (2007) (we have denoted it  $\varphi^{PN}$ ) and is called, in De Clippel and Serrano (2008), the *externality-free value*. It is viewed as a reference point rather than as an actual final recommendation of the payoffs for the players.

Skibski, Michalak, and Wooldridge (2013)<sup>13</sup> take a more direct approach and provide a

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<sup>13</sup>Skibski, Michalak, and Wooldridge (2017) provide a more condensed presentation of this value in

direct link between marginal contributions and values for PFF games. The marginal contribution of a player  $i$  to coalition  $S$  in a partition  $P$  in a game  $v$ , denoted by  $mc_i^\alpha(v)(S, P)$ , is taken to be a weighted average of  $i$ 's total effects to coalition  $S$  over  $\mathcal{P}_S$ . More formally,

$$mc_i^\alpha(v)(S, P) = \sum_{T \in P \setminus \{S\}} \alpha_i(S \setminus \{i\}, T, P) (v(S, P) - v(S \setminus \{i\}, P \setminus \{S, T\} \cup \{S \setminus \{i\}, T \cup \{i\}\}))$$

$\alpha_i(S \setminus \{i\}, T, P)$  is the weight attached to the effect on the value of  $S$ , of having player  $i$  leave  $S$  and join another partition  $T \in P$ .

A player  $i$  is an  $\alpha$ -null player in a game  $v$  if  $mc_i^\alpha(v)(S, P) = 0$  for all  $(S, P) \in ECL$  with  $i \in S$ . Then Skibski, Michalak, and Wooldridge (2013) show there is a unique value  $\varphi^{SMW}$  on  $\mathcal{G}^N$  that satisfies the standard axioms of efficiency, symmetry, additivity, together with the following axiom:

**Null player axiom $^\alpha$ .** A value  $\varphi$  satisfies the null player axiom $^\alpha$  if  $\varphi_i(v) = 0$  for any game  $v \in \mathcal{G}^N$  and any  $\alpha$ -null player  $i \in N$ .

The closed-form expression for the value  $\varphi^{SMW}$ , similar to the Shapley value for CFF games, is

$$\varphi_i^{SMW}(v) = \frac{1}{n!} \sum_{\sigma \in \Omega(N)} \sum_{P \in \mathcal{P}(N)} pr_\sigma^\alpha(P) (v(C_i^\sigma \cup \{i\}, P_{-(C_i^\sigma \cup \{i\})}) - v(C_i^\sigma, P_{-C_i^\sigma})) ,$$

where  $\Omega(N)$  denotes the set of permutations of  $N$ ,  $C_i^\sigma$  denotes the set of players before player  $i$  in the permutation  $\sigma$ ,  $pr_\sigma^\alpha(P)$  is  $\prod_{i \in N} \alpha_i(C_i^\sigma, P_{-C_i^\sigma})$  and  $P_{-S}$  is defined as in the expression for  $\varphi^{HY}$ .

It is worth noting that, in the same way as  $\varphi^{SMW}$ , several values derived axiomatically  $(\varphi^B, \varphi^{AAR}, \varphi^{MPW}, \varphi^{PN}, \varphi^{MQ}, \varphi^{HY})$  also have an interpretation as an average of suitably defined marginal contributions.

## 5 Other approaches

### 5.1 The potential approach

Hart and Mas Colell (1989) introduce the concept of a *potential* function,  $p$ . This function associates with each CFF game  $(N, \hat{v})$  a single number,  $p(N, \hat{v})$ , the potential of the game.

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section 4.2.

The marginal contribution of player  $i \in N$  to the game  $(N, \hat{v})$ , denoted by  $D^i(N, \hat{v})$ , is then defined as  $p(N, \hat{v}) - p(N \setminus \{i\}, \hat{v})$ , where the game  $(N \setminus \{i\}, \hat{v})$  is the CFF game given by the restriction of  $\hat{v}$  to  $N \setminus \{i\}$ . Furthermore, for any CFF game  $(N, \hat{v})$  the sum of the marginal contributions of the players equals  $\hat{v}(N)$ . That is,  $\sum_{i \in N} D^i(N, \hat{v}) = \hat{v}(N)$  for any CFF game  $(N, \hat{v})$ . Hart and Mas Colell (1989) show that such a function exists and the marginal contribution of each player is precisely its Shapley value.

In addition to providing a new and exciting way to look at the Shapley value as a marginal contribution, the potential concept leads to a consistency property characterization of the Shapley value. Given a game  $(N, \hat{v})$  and a value for CFF games  $\psi$ , let us define the “reduced” CFF game  $(T, \hat{v}_T^\psi)$  by:

$$\hat{v}_T^\psi(S) = \hat{v}(S \cup (N \setminus T)) - \sum_{i \in N \setminus T} \psi^i(S \cup (N \setminus T), \hat{v}) \text{ for all } S \subset T$$

$(S \cup (N \setminus T), \hat{v})$  is the game  $(N, \hat{v})$  restricted to  $S \cup (N \setminus T)$ . A value  $\psi$  is consistent if  $\psi^j(T, \hat{v}_T^\psi) = \psi^j(N, \hat{v})$  for any CFF game  $(N, \hat{v})$ , any  $T \subset N$ , and any  $j \in T$ . Hart and Mas Colell (1989) show that a value  $\psi$  is consistent and “equally splits the surplus” for two-person games if and only if it is the Shapley value.

Dutta, Ehlers, and Kar (2010) extend the potential notion to PFF games by defining restriction operators that quantify the marginal contribution of a player  $i \in N$  to a game  $v \in \mathcal{G}^N$ . A restriction operator  $r$  associates with each game  $(N, v)$  and each player  $i \in N$  a subgame  $(N \setminus \{i\}, v^{-i,r})$ . The worth  $v^{-i,r}(S, P)$  of an embedded coalition  $(S, P) \in ECL(N \setminus \{i\})$  is a function, implicit in the definition of the mapping  $r$ , of the values  $v(S, P')$ , where  $P'$  is any partition that can arise from partition  $P$  by adding player  $i$  (player  $i$  may enter as a singleton or join one of the existing coalitions in  $P$ ). This definition imposes very little structure on the subgames. Dutta, Ehlers, and Kar (2010) start by requiring that the restriction operators satisfy path independence. To introduce the assumption, let  $v^{-ij,r} = v^{-i,r}(v^{-j,r})$ .

**Path independence axiom.** A restriction operator satisfies the path independence axiom if  $v^{-ij,r} = v^{-ji,r}$ .

That is, the order by which players are removed does not affect the game taking place after their departure.

Given a restriction operator  $r$  satisfying path independence, an  $r$ -potential function,  $p^r : \mathcal{G}^N \rightarrow \mathcal{R}$ , is similarly defined to the potential definition in CFF games. Marginal contributions of players are given by  $D^i p^r(N, v) = p^r(N, v) - p^r(N \setminus \{i\}, v^{-i,r})$  for all  $i \in N$  and they sum up to  $v(N, \{N\})$ . Each potential function  $p^r$  gives rise to what Dutta, Ehlers, and Kar (2010) call an  $r$ -Shapley value.

Still, there are several  $r$ -Shapley values. For example,  $\varphi^{PN}$  (the externality-free value of De Clippel and Serrano, 2008) is obtained by letting  $v^{-i,r}(S, P) = v(S, (P \cup \{i\}))$ .<sup>14</sup> The value  $\varphi^{AAR}$  is obtained when  $v^{-i,r}(S, P)$  is a weighted average of the  $v(S, P')$ 's ( $P'$  is again any partition that can arise from partition  $P$  by adding player  $i$ ).<sup>15</sup>

Imposing further axioms on the restriction operators singles out particular families of values for PFF games. Furthermore, Dutta, Ehlers, and Kar (2010) study the relationship between the axioms on the restriction operators and the extension of the standard Shapley axioms to PFF games. The restriction operators also enable the authors, similar to Hart and Mas Colell (1989), to define a consistency property for PFF games. They show under some further assumptions that the unique value satisfying consistency for a given restriction operator  $r$  is the  $r$ -Shapley value.

## 5.2 The Harsanyi dividends approach

Another approach that leads to the Shapley value involves the use of “dividends.” For any CFF game  $(N, \hat{v}) \in G^N$ , the dividends that a coalition  $S$  generates are recursively defined as follows:

$$\Delta_{\hat{v}}(S) = \begin{cases} 0 & \text{if } S = \emptyset \\ \hat{v}(S) - \sum_{T \subset S, T \neq S} \Delta_{\hat{v}}(T) & \text{if } S \neq \emptyset \end{cases}.$$

Harsanyi (1959) proves that the Shapley value evenly distributes the dividends of each coalition to the players comprising it. That is,  $\psi_i^{Sh}(N, \hat{v}) = \sum_{S \subset N, i \in S} \frac{1}{|S|} \Delta_{\hat{v}}(S)$ .

Macho-Stadler, Pérez-Castrillo, and Wettstein (2010) show that a similar construction leads to any value  $\varphi$  for PFF games that is constructed through the average approach with weights  $\alpha(S, P)$ . The dividends for any embedded coalition  $(S, P)$  are defined recursively

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<sup>14</sup>Dutta, Ehlers, and Kar (2010) call it the *sing restriction operator*.

<sup>15</sup>See Dutta, Ehlers, and Kar (2010) for the full description of the weighting system used.

as follows:

$$\Delta_v^\alpha(S, P) = \begin{cases} 0 & \text{if } S = \emptyset \\ v(S, P) - \sum_{(T, Q) \in ECL, T \subset S, T \neq S} \alpha(T, Q) \Delta_v^\alpha(T, Q) & \text{if } S \neq \emptyset. \end{cases}$$

That is, dividends received by subsets of  $S$  are all taken into account scaled down in accordance to the weights associated with each partition. As in the CFF case, the value for player  $i$ ,  $\varphi_i^\alpha(N, v)$  can be expressed as:

$$\varphi_i^\alpha(N, v) = \sum_{\substack{(S, P) \in ECL \\ S \ni i}} \frac{1}{|S|} \alpha(S, P) \Delta_v^\alpha(S, P).$$

That is, dividends, taking into account the embedded coalition generating them, are equally shared among the players comprising the embedded coalition.<sup>16</sup>

### 5.3 Algorithms

One of the most popular interpretations of the Shapley value in CFF games, already present in Shapley's thesis (Shapley, 1953b), is that the value of a player can be computed using the  $n!$  orders in which the players can arrive to the game: the Shapley value of the player is his average marginal contribution in a sequential process where each order has the same probability of happening.

Skibski, Michalak, and Wooldridge (2017) extend this interpretation to PFF games. They envision a situation where the partition that a player encounters and the coalition that he joins when he leaves a coalition is the result of the "Chinese restaurant process," where players are sequentially assigned to coalitions; the  $k$ -th player (except for the first one) is assigned to a coalition in proportion to the size of that coalition, and he remains single with probability  $1/k$ . Thus, the marginal contribution of a player to a coalition (or, equivalently, the contribution that the coalition loses when the player leaves) is computed as the average of the contributions for all the possible coalitions and partitions that can emerge from the Chinese restaurant process. Skibski, Michalak, and Wooldridge (2017) define the stochastic Shapley value  $\varphi^{SMW}$  as the average of the average (according to the

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<sup>16</sup>Modifying the summation of dividends by introducing a vector of player weights, Macho-Stadler, Pérez-Castrillo, and Wettstein (2010) obtain a weighted Shapley value for games with externalities.

previous process) marginal contributions of each player when each permutation has the same probability of happening.

The stochastic Shapley value can be characterized as the unique value that satisfies efficiency, symmetry, additivity, and the CRP-null player axiom. A player is a CRP-null player if his marginal average contribution (where the average is again computed using the Chinese restaurant process) is zero. The CRP-null player axiom requires that his payoff is zero. The stochastic Shapley value coincides with the value proposed by Feldman (1996) and Macho-Stadler, Pérez-Castrillo, and Wettstein (2007), that is,  $\varphi^{SMW} = \varphi^{MPW}$ .

## 6 Non-cooperative approaches to value extensions

### 6.1 Implementation

Values for cooperative games are often viewed as a recommendation of how to share jointly earned profits. A natural question regarding cooperative solutions is whether they can be implemented. In other words: Can a designer, who does not know the CFF or PFF game the agents are facing, design a game-form (a mechanism) leading in equilibrium to the payoffs recommended by the solution?

This question was positively answered for the Shapley value for CFF games. Winter (1994) and Dasgupta and Chiu (1998) propose demand commitment games in which, for some uniformly chosen random order of the players, each player can either make a demand to the following player or form a coalition satisfying the demands of some of the preceding players. For strictly-convex CFF games, these mechanisms implement the Shapley value in expectation, that is, the expected payoff of every player (over all possible orderings) coincides with his Shapley value. Pérez-Castrillo and Wettstein (2001 and 2002) construct bidding mechanisms, where players compete for the right to make a proposal to other players, that implement the Shapley value directly, and not just in expectation, for zero-monotonic CFF games. A CFF game  $(N, \hat{v})$  is zero-monotonic if  $v(S) + v(i) \leq v(S \cup \{i\})$  for any subset  $S \subseteq N$  and any  $i \notin S$ .

Macho-Stadler, Pérez-Castrillo, and Wettstein (2006) generalize these mechanisms to

games with externalities by adding a coalition(partition)-forming stage. They construct two mechanisms implementing solution concepts derived through the average approach. One mechanism is designed for environments with positive externalities and the other for environments with negative externalities. A PFF game  $(N, v)$  has negative externalities if  $v(S, P) \geq v(S, P')$  for every  $P, P'$ , when each element in  $P'$  is given by a union of elements in  $P$ , that is,  $P$  is a refinement of  $P'$ . A PFF game  $(N, v)$  has positive externalities if  $v(S, P) \leq v(S, P')$  for every  $P, P'$ , where  $P$  is a refinement of  $P'$ .

Similarly, Ju and Wettstein (2008) construct a mechanism implementing  $\varphi^{PN}$  through a different generalization of the bidding mechanisms introduced in Pérez-Castrillo and Wettstein (2001) (see also Ju and Wettstein, 2009).

## 6.2 A bargaining approach

Another common support for values is given by providing reasonable or attractive bargaining procedures realizing them. Note that unlike the implementation approach, it is assumed that promises in utility terms can be enforced or, alternately, are truthfully carried out. Gul (1989 and 1999) and Hart and Levi (1999) provide bargaining protocols with pairwise meetings that under some conditions on the underlying CFF game (strict convexity or strict super-additivity) lead to expected payoffs coinciding with Shapley value payoffs. Hart and Mas-Colell (1996) construct a bargaining protocol with multilateral meetings leading in expectation to the Shapley value payoffs for CFF games and the Nash bargaining solution for pure bargaining problems.

McQuillin (2009) shows that a simple adaptation of Gul's (1989) protocol leads to  $\varphi^{MQ}$ . Also, McQuillin and Sugden (2016) construct another finite bargaining process, the deadline bargaining game, which for PFF games with negative externalities leads again to  $\varphi^{MQ}$ . The deadline bargaining game assumes the same form as Gul's (1989) bargaining in each period, except for the final period where each active player receives the value of the coalition he represents.

Grabisch and Funaki (2012) propose three values for PFF games, each corresponding to a distinct procedure of coalition formation. The values are different from the values

suggested thus far in this chapter as they do not match the Shapley value for PFF games that are CFF games. Grabisch and Funaki (2012) do suggest modified values that reduce to the Shapley value. However, they argue that “pure” coalition formation values should not reduce to the Shapley value, since in the coalition formation scenarios all players are always “present in the game” whereas in the Shapley value there is a distinction based on the order in which players arrive.

Maskin (2003), in his Presidential Address to the Econometric Society, studies cooperation in the presence of externalities using a set of bargaining procedures, where all orderings of the players are possible at the offset. He draws a clear distinction between environments with negative and positive externalities. He then stresses that in the presence of positive externalities the assumption that the grand coalition forms, even if it is efficient, is problematic and may not be supported by any reasonable bargaining procedure. Several axioms are formulated regarding the bargaining procedures and the payoffs they generate at the various stages. These axioms are satisfied by several sharing schemes, which form a family of generalized Shapley values. These values determine both which coalitions form and how the surplus is shared among their members.

Borm, Ju, and Wettstein (2015) also take a bargaining perspective to analyze PFF games. They use a sequential approach to calculate the “reasonable” worth of any coalition (when in reality the worth depends on the whole partition) so that the Shapley value can be used to identify the value of each player in the game. To calculate the worth of a coalition  $S \subseteq N$ , Borm, Ju, and Wettstein (2015) envision a process where coalition  $S$  “moves first” by forming a coalition structure within itself, taking into account that the members of  $N \setminus S$  would choose a partition that maximizes the value of  $N \setminus S$  (and if there is more than one such partition, the one chosen is the most detrimental to  $S$ ). Bearing that in mind, the members of  $S$  choose the coalition structure that maximizes their terminal payoff. Once the worth  $\hat{v}(S)$  of a coalition is constructed in this way, they define the rational belief Shapley value as the Shapley value of the game  $(N, \hat{v})$ , that is,  $\varphi^{BJW}(N, v) = \psi^{Sh}(N, \hat{v})$ . Borm, Ju, and Wettstein (2015) also propose variations of the sequential approach, leading to two further values, and provide mechanisms that share a

common bargaining structure and implement the three values.

## 7 Conclusion

In this chapter, we have reviewed several extensions of the Shapley value for environments where externalities among coalitions are present. The various approaches that lead to the Shapley value in characteristic function form games (axiomatic, marginalistic, potential, dividends, algorithmic, and non-cooperative) have provided alternative routes to address the question of the most suitable extension of this value for the larger class of games in partition function form. It is worth noting that some of the proposed values emerge from, and can thus be supported through, all or most of the previous approaches.

The main reason to study cooperative solution concepts for games with externalities is that the existence of externalities is the rule rather than the exception in most interesting environments. Therefore, the extensions that we have reviewed should be of interest to researchers looking for solution concepts in such environments.

Interestingly, some of these values have already been applied for studying competitive markets and environmental agreements, both natural fields for applying extensions of the Shapley value for games with externalities. For example, Jelnov and Tauman (2009) consider a game in coalitional form played by the firms in a Cournot industry and an outside innovator who owns a cost-reducing innovation. The firms can form at most two coalitions: the coalition including the innovator and some firms (that will use the new technology in their productio processes), and the complementary coalition of firms. Using the Feldman's (1996) and Macho-Stadler, Pérez-Castrillo, and Wettstein's (2007) extension of the Shapley value, Jelnov and Tauman (2009) show that when the industry size goes to infinity, the Shapley value of the innovator approximates the payoff he obtains in a standard non-cooperative setup where he has the entire bargaining power. Another example is provided by Liu, Lindroos, and Sandal (2016) who study both cooperative and competitive solutions for managing a fish stock. In a three-country environment, taking the Norwegian spring-spawning herring as a case study, they analyze the stability of the grand coalition in a rich harvest model where the catch function is density-dependent.

In their model, players (Norway, Russia, and the remaining countries fishing there) are asymmetric and, when they cooperate, share the benefits according to the “externality-free” Shapley value introduced by Pham Do and Norde (2007). Their conclusion is that the likelihood of a stable grand coalition increases with the degree of asymmetry in the players’ efficiency levels.

The values analyzed in this chapter aim at providing a solution concept that can be applied in any environment where externalities among coalitions exist, independently of the type of externality. Still, we know that the externalities present in some environments are positive (think of the environmental coalitions) whereas they are negative in other situations (as is the case for trading agreements). Some of the solution concepts studied in this chapter may be better suited to some types of externalities than to others. Moreover, it might be advisable to consider extensions of the Shapley value that are suitable just for a subset (for example, the subset of games with positive externalities, or yet a smaller subset where all positive externalities have the same worth) of PFF games. Depending on the features of the subset, it may be possible to propose new axioms, reflecting properties that are desirable for the type of externalities considered, that characterize extensions of the Shapley value well-suited to these environments.

From the opposite point of view, it may be worthwhile extending some of the ideas developed in this chapter to sets larger than the set of partition function form games. Indeed, some environments are characterized by the presence of externalities not only across coalitions but also across issues that are linked in the sense that the worth of a coalition in one issue depends on the organization of the players on all the issues. Consider countries negotiating both a trade agreement and an environmental agreement. On this occasion these two issues, trade and environment, are linked. In particular, the accelerated growth triggered by a trade liberalization if countries form a large coalition is likely to raise CO<sub>2</sub> emissions, making it more difficult for the participants in an environmental agreement to comply with their obligations. Therefore, the worth of a coalition on trade depends on the partition of the countries following an environmental negotiation. A first attempt in this direction is Diamantoudi, Macho-Stadler, Pérez-Castrillo, and Xue (2015)

who extend values for partition function form games (that also satisfy the strong dummy property) to environments where externalities across issues are present.

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