GAME THEORY
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Problem Set 3
(to be returned on Tuesday, September 26th)

EXERCISE 1. (Voting by alternating veto) Two people select an action that affects them both by alternately vetoing actions until only one remains. Suppose there are three possible actions, X, Y, and Z, person 1 is the first to move, person 1 prefers X to Y to Z, and person 2 prefers Z to Y to X.

(a) Model this situation as an extensive game and find its Nash equilibria.

(b) Find the subgame perfect equilibria. Does the game have any Nash equilibrium that is not a subgame perfect equilibrium? Is any outcome generated by a Nash equilibrium not generated by any subgame perfect equilibrium? Consider variants of the game in which player 2's preferences may be different from those specified previously. Are there any preferences for which the outcome in a subgame perfect equilibrium of the game in which player 1 moves first differs from the outcome in a subgame perfect equilibrium of the game in which player 2 moves first?

EXERCISE 2. (Dollar auction) An object that two people each value at v (a positive integer) is sold in an auction. In the auction, the people alternately have the opportunity to bid; a bid must be a positive integer greater than the previous bid. (In the situation that gives the game its name, v is 100 cents.) On her turn, a player may pass rather than bid, in which case the game ends and the other player receives the object; *both* players pay their last bids (if any). (If player 1 passes initially, for example, player 2 receives the object and makes no payment; if player 1 bids 1, player 2 bids 3, and then player 1 passes, player 2 obtains the object and pays 3, and player 1 pays 1.) Each person's wealth is w, which exceeds v; neither player may bid more than her wealth. For $v = 2$ and $w = 3$ model the auction as an extensive game and find its subgame perfect equilibria.

EXERCISE 3. (Comparing simultaneous and sequential games) The set of actions available to player 1 is A_1; the set available to player 2 is A_2. Player l's preferences over pairs (a_1, a_2) are represented by the payoff $u_1(a_1, a_2)$, and player 2's preferences are represented by the payoff $u_2(a_1, a_2)$. Compare the Nash equilibria (in pure strategies) of the strategic game in which the players choose actions simultaneously with the subgame perfect equilibria of the extensive game in which player 1 chooses an action, then player 2 does so. (That is, for each history a_1, the set of actions available to player 2 is A_2.)

(a) Show that if, for every value of a_1, there is a unique member of A_2 that maximizes $u_2(a_1, a_2)$, then in every subgame perfect equilibrium of the extensive game, player l's payoff is at least equal to her highest payoff in any Nash equilibrium of the strategic game.

(b) Show that it is possible that player 2's payoff in every subgame perfect equilibrium of the extensive game is higher than her highest payoff in any Nash equilibrium of the strategic game.

(c) Show that if for some values of a_1 more than one member of A_2 maximizes $u_2(a_1, a_2)$, then the extensive game may have a subgame perfect equilibrium in which player l's payoff is less than her payoff in all Nash equilibria of the strategic game.

EXERCISE 4. Let G be a two-player strategic game $(\{1, 2\}, (A_i), (u_i))$ in which each player has two actions: $A_i = \{a_i', a_i''\}$ for $i = 1, 2$. Show that G is the strategic form of an extensive game with perfect information if and only if either for some a_1 in A_1 we have $u_i(a_1, a_2') = u_i(a_1, a_2'')$ for $i = 1, 2$ or for some a_2 in A_2 we have $u_i(a_1', a_2) = u_i(a_1'', a_2)$ for $i = 1, 2$.

EXERCISE 5. Give an example of an infinite horizon game for which the one deviation property does not hold.

EXERCISE 6. Show that the requirement in Kuhn's theorem that the game be finite cannot be replaced by the requirement that it have a finite horizon.

EXERCISE 7. Say that a finite extensive game with perfect information satisfies the *no indifference condition* if

$$u_j(z) = u_j(z') \text{ whenever } u_i(z) = u_i(z') \text{ for some } i \text{ in } N,$$

where z and z' are terminal histories. Show, using induction on the length of subgames, that every player is indifferent among all subgame perfect equilibrium outcomes of such a game. Show also that if s and s' are subgame perfect equilibria then so is s'', where for each player i the strategy s_i'' is equal to either s_i or s_i' (i.e., the equilibria of the game are *interchangeable*).