

# Self-Selection Consistent Functions\*

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**Abstract.**

This paper studies collective choice rules whose outcomes consist of a collection of simultaneous decisions, each one of which is the only concern of some group of individuals in society. The need for such rules arises in different contexts, including the establishment of jurisdictions, the location of multiple public facilities, or the election of representative committees. We define a notion of allocation consistency requiring that each partial aspect of the global decision taken by society as a whole should be ratified by the group of agents who are directly concerned with this particular aspect. We investigate the possibility of designing efficient allocation consistent rules. We also explore whether such rules may in addition respect the Condorcet criterion.

**Key words:** Social Choice Rules, Allocation Consistency, Condorcet criterion, Participation.

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## 1. Introduction.

Collective choices often involve multiple simultaneous decisions, whose particular aspects may affect different agents to different degrees. If new borders are drawn in a region of the world, I am mainly affected by what my country will look like, although I may also care about the whole map of the region. If a committee is chosen to negotiate on behalf of my union, I am especially interested on those delegates that I am acquainted with, and/or who will more closely represent my interest.

In this paper we concentrate on polar cases, where each agent is solely concerned with one of the components of the global decision, and congestion effects are ignored. For example, several hospitals may be built simultaneously, but if each agent is only allowed to use one of them (and congestion levels are similar), then he will essentially evaluate the overall decision in terms of the particular hospital he is assigned to. Under these circumstances, we discuss the merits of different social choice procedures to determine (1) what set of objects should be chosen, and (2) which agents should benefit from each of the objects. Each procedure is defined to give answers to both questions, for each possible set of agents and for any number of objects to be chosen.

Since we consider that agents are assigned to specific objects, and that they only care about them, an interesting question arises regarding the overall consistency of the collective procedure. Once a decision is taken, all agents who share the same object emerge naturally as a meaningful group. All those citizens of a new nation after border redrawing, all the trade union members whose opinion will be channeled by a given representative, all users of a new hospital are concerned about the same aspects of the global decision. What if they challenge the global decision by suggesting that, as far as they are concerned, the partic-

ular object that they have been assigned to should be changed for another one? What if all people who, given the public decision to build hospitals  $H_1, H_2$ , are assigned to  $H_1$ , then demand that  $H'_1$  be built instead? What if, after talking to one delegation member, the agents he is supposed to represent meet and vote in favor of substituting him for somebody else? In all these cases, there would be some inconsistency between a global decision which turns a group into the major beneficiary of one of its aspects, and the partial decision that these same concerned agents would suggest, regarding this particular aspect. This would be a definite problem if one of these groups is not only interested in changing the partial decision, but is actually entitled to do so, because the rule assigns a different single decision to the group when only one has to be chosen. Social choice procedures which avoid these problems will be called *allocation consistent*.

Many authors have been concerned about the connections among different decisions taken by societies when their members or their resources vary. Different conditions have been imposed requiring that the changes in the social decision associated with changes in the membership of society, or with changes in the set of possible outcomes, respect some notion of consistency (see for example Thomson [9] for a survey on consistency).

Our concern can also be viewed as one of consistency, but we must qualify the analogy. We want to emphasize the fact that our consistency requirement refers to the connections between global choices and their particular aspects: on this account, our focus is restricted, since we only consider models where this distinction makes sense. A second difference is, we believe, in favor of our notion. We do not look at exogenous changes in the membership of society, which may or may not be reasonably expected. We concentrate on the connections between global decisions, taken by the society at large, and their partial components, as

viewed by those agents in the very same society who are affected and concerned by those partial aspects of the decision. For those problems where the structure of the global decision is naturally decomposable, and agents are particularly concerned with only parts of the global picture, we find our notion of allocation consistency to be particularly attractive.

In this paper, we focus on rules which are not only allocation consistent, but also efficient. In addition to its usual justification, efficiency is a particularly interesting requirement in our context, because efficient decisions will also be envy-free. Allocation rules which are allocation consistent and efficient are called *self-selection consistent*, to emphasize the idea that the concerned agents self-select themselves to play this role, through their voluntary identification with one of the projects, hospitals, representatives or nations.

In order to discuss these general issues in a specific context, we concentrate on a particular setup. Formally, we study the problem of choosing  $k$  objects on the real line, when the preferences of agents over single objects are single peaked.

We first prove that self-selection consistent procedures can be defined. In fact, we present a systematic procedure to create them, starting from simple rules which only determine a single choice for each group. We show that if the simple rule satisfies a condition of participation, then a self-selection consistent rule can be defined for any set of agents and any number of partial decisions. Moreover, this general rule will recommend the same partial decisions that would be made, under the initial simple rule, by each of the groups that self selects themselves as interested in each one of them, if the group alone was called to make a one-element decision. In this sense, the definition of the general rule can be interpreted as an extension of the simple one, and our result is that we can extend any simple rule satisfying participation to create self-selection consistent

general rules. We also offer a partial converse of the preceding result. Start from a self selection consistent general rule and consider its restriction to the cases where groups are only called to make one simple decision. If the general rule satisfies an additional condition of simplicity, then its associated simple rule must satisfy participation. Finally, we investigate the extent to which self-selection consistent rules can be made compatible with the Condorcet principle, a requirement which is often considered a must for a collective choice rule to be well behaved. We show that there is no problem in accommodating the Condorcet principle for simple decisions, involving the choice of one object only. But we also prove that no general rule can be self-selection consistent and respect the Condorcet principle, when the rule is a function defined for all sets of voters and any number of simultaneous decisions. This unwelcome impossibility result must be qualified: it could be overcome if we allowed our rules to be somewhat undecisive, by considering correspondences instead of functions (Barberà and Beviá [2]).

Setups where several points are chosen on a line and agents cluster around these points arise naturally in different parts of the economic literature and they admit several interpretations. They provide the basic model for the analysis of local public goods and jurisdictional questions (see Alesina and Spolaore [1], Greenberg and Weber [3], Jehiel and Schotchmer [4], Konishi et al., [5], Milchtaich and Winter [6], Tiebout [10]). Our specific model is particularly explicit about the connections between the global decision of the whole group and the partial decisions of its different subgroups; our main focus is on allocation consistency. This is done at some cost. We explicitly rule out congestion effects, which are important in many contexts. We also take the number of objects to be chosen as an exogenous parameter (in contrast with models where the number of jurisdictions is an endogenous variable). These two restrictive features of our model

are borrowed from a series of recent papers by Miyagawa [7]. His model is very similar to ours, but we have expanded it to encompass the possibility of a variable electorate to choose a variable number of objects. This extension allows us to stress the issue of consistency and the endogenous character of the groups that share each single object. Even if our models are similar, Miyagawa's analysis and conclusions are very different from ours. His choice of axioms leads him to characterize different rules which tend to select rather extreme outcomes. Moreover, his formal analysis often stops at the case where only two objects are chosen. In contrast, our analysis applies to any fixed number of partial choices.

The rest of the paper is organized as follows. In Section 2, we present our model in detail. Section 3 presents the main results of the paper regarding existence and characterization of self-selection consistent rules. Section 4 studies the existence of self-selection consistent rules respecting the Condorcet criterion. Finally, in Section 5 we conclude.

## 2. The Model.

We consider problems that involve any finite set of agents. Agents are identified with elements in  $\mathbb{N}$ , the set of natural numbers. Let  $\mathfrak{S}$  be the class of all finite subsets of  $\mathbb{N}$ . Elements of  $\mathfrak{S}$ , denoted as  $S, S', \dots$ , stand for particular societies, whose cardinality is denoted by  $|S|, |S'|$ , etc.

We now describe the decisions that societies can face. These are determined by the number and the position of relevant locations, and by the sets of agents who are allocated to each location.

A natural number  $k \in \mathbb{N}$  will stand for the number of locations. Then, given  $S \in \mathfrak{S}$  and  $k \in \mathbb{N}$ , an  $S/k$ -decision is a  $k$ -tuple of pairs  $d = (x_h, S_h)_{h=1}^k$ , where  $x_h \in \mathbb{R}$ , and  $(S_1, \dots, S_k)$  is a partition of  $S$ . We interpret each  $x_h$  as a location and  $S_h$  as the set of agents who is assigned to the location  $x_h$ . Notice that elements in the partition may be empty. This will be the case, necessarily, if  $k > |S|$ . We call  $d_L = (x_1, \dots, x_k)$  the vector of *locations*, and  $d_A = (S_1, \dots, S_k)$  the vector of *assignments*.

Let  $D(S, k)$  be the set of  $S/k$ -decisions,  $D(k) = \bigcup_{S \in \mathfrak{S}} D(S, k)$  the set of  $k$ -decisions, and  $D = \bigcup_{k \in \mathbb{N}} D(k)$  the set of decisions. For each agent  $j \in \mathbb{N}$ , the set of  $k$ -decisions which concern  $j$  is  $D_j(k) = \bigcup_{\{S \in \mathfrak{S} | j \in S\}} D(S, k)$  and the set of decisions that concern  $j$  is  $D_j = \bigcup_{k \in \mathbb{N}} D_j(k)$ .

Agents are assumed to have complete, reflexive, transitive preferences over decisions which concern them. That is, *agent  $i$ 's preferences* are defined on  $D_i$ , and thus, rank any pair of  $S/k$  and  $S'/k'$ -decisions provided that  $i \in S \cap S'$ . Denote by  $\succsim_i$  the preferences of agent  $i$  on  $D_i$ .

We shall assume all along that preferences are *singleton-based*. Informally, this means that agents' rankings of decisions only depend on the location they are assigned to, not on the rest of locations or on the assignment of other agents

to locations. This assumption is compatible with our interpretation that agents can only use the good provided at one location, and that this is a public good subject to no congestion. Formally, a preference  $\succsim_i$  on  $D_i$  is singleton-based if there is a preference  $\bar{\succsim}_i$  on  $\mathbb{R}$  such that for all  $d, d' \in D_i$ ,  $d \succsim_i d'$  if and only if  $x(i, d) \bar{\succsim}_i x(i, d')$ , where  $x(i, d)$  denotes the location to which agent  $i$  is assigned under the decision  $d$ .

In all that follows, we shall assume that for all  $i \in N$ ,  $\succsim_i$  is singleton-based, and in addition, that the order  $\bar{\succsim}_i$  is *single-peaked*. That is: for each  $\bar{\succsim}_i$ , there is an alternative  $p(i)$  which is the unique best element for  $\bar{\succsim}_i$ ; moreover, for all  $x, y$ , if  $p(i) \geq x > y$ , then  $x \bar{\succsim}_i y$ , and if  $y > x \geq p(i)$ , then  $x \bar{\succsim}_i y$ . Under the singleton-based assumption, there is a one to one relation between preferences on decisions,  $\succsim_i$ , and preferences on locations,  $\bar{\succsim}_i$ , thus, from now on we will not make any distinction among them.

Given  $S \in \mathfrak{S}$ , *preference profiles for  $S$*  are  $|S|$ -tuples of preferences, and we denote them by  $P_S, P'_S, \dots$

We denote by  $\mathcal{P}$  the set of all preferences described above, and by  $\mathcal{P}^S$  the set of preference profiles for  $S$  satisfying those requirements.

A *collective choice function* will select a  $k$ -decision, for each given  $k$ , on the basis of the preferences of agents in  $S$ , for any  $S \in \mathfrak{S}$ . Formally,

**Definition 1.** A *collective choice function* is a function  $\varphi : \bigcup_{S \in \mathfrak{S}} \mathcal{P}^S \times \mathbb{N} \rightarrow D$  such that, for all  $S \in \mathfrak{S}$ ,  $P_S \in \mathcal{P}^S$  and  $k \in \mathbb{N}$ ,  $\varphi(P_S, k) \in D(S, k)$ .

We now propose natural and attractive properties that a collective choice function may or may not satisfy. The first, allocation consistency, is proposed here for the first time. It reflects the desire for consistency between global decisions and their partial components that we have already discussed at length in our introduction.

**Definition 2.** A collective choice function  $\varphi$  is allocation consistent if for all  $S \in \mathfrak{S}$ ,  $P_S \in \mathcal{P}^S$  and  $k \in \mathbb{N}$ , if  $((x_1, S_1), \dots, (x_k, S_k)) = \varphi(P_S, k)$ , then  $(x_h, S_h) = \varphi(P_{S_h}, 1)$  for all  $h$  such that  $S_h \neq \emptyset$ .<sup>1</sup>

In order to fix ideas, we provide two examples of allocation consistent rules. In both examples, locations coincide with the position of the peaks of the  $k$  agents with the smallest index (with some special rules when different agents have the same peak; see the example for details). In Example 1, agents are assigned to locations in a very arbitrary way, while in Example 2 agents are allocated to the location that they would prefer among the existing ones. Comparing these two allocation consistent rules will allow us to discuss the importance of imposing, in addition to this property, other desiderata like efficiency and no-envy.

**Example 1.** Let the set of potential agents be the set of integers, endowed with the natural order. For any  $S = \{i_1, \dots, i_H\} \in \mathfrak{S}$ ,  $P_S \in \mathcal{P}^S$  and  $k \in \mathbb{N}$ , define the  $S/k$ -decision  $\varphi(P_S, k) = (x_h, S_h)_{h=1}^k$  as follows. Let

$$x_1 = p(i_{l^1}) \equiv p(i_1)$$

$x_2 = p(i_{l^2})$ , where  $l^2$  is the smallest integer in  $S$  such that  $p(i_{l^2}) \neq p(i_{l^1})$  if it exists. Otherwise  $x_2 = \dots = x_k = x_1$ .

$x_3 = p(i_{l^3})$ , where  $l^3$  is the smallest integer in  $S$  such that  $p(i_{l^3}) \neq p(i_{l^h})$  for all  $h < 3$  if it exists. Otherwise  $x_3 = \dots = x_k = x_2$ .

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$x_k = p(i_{l^k})$ , where  $l^k$  is the smallest integer in  $S$  such that  $p(i_{l^k}) \neq p(i_{l^h})$  for all  $h < k$  if it exists. Otherwise  $x_k = x_{k-1}$ .

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<sup>1</sup>Since for  $k = 1$  there is a unique assignment of the agents (all of them together), we will often abuse notation and write  $x_h = \varphi(P_{S_h}, 1)$  when  $(x_h, S_h) = \varphi(P_{S_h}, 1)$ .

Let  $m$  be the highest value of  $h \leq k$  such that  $l^h$  is well defined. Now let  $S_1 = \{i_h \in S \mid p(i_h) = p(i_{l^1})\}$ ,  $S_2 = \{i_h \in S \mid p(i_h) = p(i_{l^2})\}$ , ...,  $S_m = \{i_h \in S \mid p(i_h) = p(i_{l^m})\} \cup (S \setminus \cup_{h=1}^{m-1} S_h)$ , and  $S_t = \emptyset$  for all  $t \in \{m+1, \dots, k\}$ . Clearly, this rule is allocation consistent.

**Example 2.** Same as above, except for the assignment of the agents. Now, let  $S_h = \{i \in S \mid x_h \succ_i x_t \text{ for all } t \in \{h+1, \dots, m\}, \text{ and } x_h \succ_i x_l \text{ for all } l \in \{1, \dots, h-1\}\}$  for all  $h \in \{1, \dots, m\}$ ,  $S_t = \emptyset$  for all  $t \in \{m+1, \dots, k\}$ . This rule is also allocation consistent.

As we have already mentioned, the difference between the above two examples lies on the assignment of the agents to locations. In Example 1 the assignment is inefficient and in Example 2 it is efficient. Notice, moreover, that under rule 2, no agent envies any other, since each one is assigned to the location he prefers.

We now proceed to define *no-envy* and *efficiency* and to study their compatibility with allocation consistency.

**Definition 3.** An  $S/k$ -decision  $d \in D(S, k)$  is *envy-free* if for all  $i \in S$ ,  $x(i, d) \succ_i x_h$  for all  $x_h \in d_L$ .

**Definition 4.** An  $S/k$ -decision  $d$  is *efficient* if there is no  $S/k$ -decision  $d'$  such that  $d' \succ_i d$  for every agent  $i \in S$  and  $d' \succ_j d$  for some  $j \in S$ .

**Definition 5.** A collective choice function  $\varphi$  is *efficient* if for all  $S \in \mathfrak{S}$ ,  $P_S \in \mathcal{P}^S$  and  $k \in \mathbb{N}$ ,  $\varphi(P_S, k)$  selects an efficient  $S/k$ -decision.

We now present the definition which appears in the title of the paper.

**Definition 6.** A collective choice function is *self-selection consistent* if it is efficient and satisfies allocation consistency. That is, for all  $S \in \mathfrak{S}$ ,  $P_S \in \mathcal{P}^S$  and

$k \in \mathbb{N}$ ,  $\varphi(P_S, k) \in \{d = (x_h, S_h)_{h=1}^k \in D(S, k) \mid d \text{ is efficient and } x_h = \varphi(P_{S_h}, 1) \text{ for all } h \text{ s.t. } S_h \neq \emptyset\}$ .

Notice that any efficient  $S/k$ -decision is envy-free. Agents in an efficient decision must be assigned to the location that they prefer. Hence, under a self-selection consistent rule, the groups of agents whose partial decisions must match with the global decision are self selected as the set of people who would attach themselves to each location, out of a voluntary choice.

### 3. Main Results

The examples in the preceding Section tell us that there exist self-selection consistent rules (Example 2), and also that some allocation consistent rules may fail to be self-selection consistent (Example 1) because they might generate envy.

In order to understand the structure of self-selection consistent rules, it is crucial to relate them to simple functions, of the form  $f : \bigcup_{S \in \mathfrak{S}} \mathcal{P}^S \rightarrow \mathbb{R}$ , which can be interpreted to determine a single location for each preference profile.

We can show that any such function  $f$  satisfying efficiency and a condition which we call *participation* can be used to define a self-selection consistent rule  $\varphi$ . This is the object of Proposition 1, which thus answers the question of existence of self-selection consistent rules in the positive and by way of a constructive proof. In fact, the function  $\varphi$  constructed with the use of  $f$  will be such that, for all  $S \in \mathfrak{S}$  and all  $P_S \in \mathcal{P}^S$ ,  $f(P_S) = \varphi(P_S, 1)$ .

We now define efficiency and *participation* and state the announced result.

**Definition 7.** A function  $f : \bigcup_{S \in \mathfrak{S}} \mathcal{P}^S \rightarrow \mathbb{R}$  is efficient if for all  $S \in \mathfrak{S}$ , and for all  $P_S \in \mathcal{P}^S$  there is no location  $x \in \mathbb{R}$  such that  $x \succ_i f(P_S)$  for all  $i \in S$  and  $x \succ_j f(P_S)$  for some  $j \in S$ .

**Definition 8.** A function  $f : \bigcup_{S \in \mathfrak{S}} \mathcal{P}^S \rightarrow \mathbb{R}$  satisfies participation if for all  $S \in \mathfrak{S}$ , all  $i \notin S$ , and for all  $(P_S, P_{\{i\}}) \in \mathcal{P}^{S \cup \{i\}}$ ,  $f(P_S, P_{\{i\}}) \succ_i f(P_S)$ .

Participation says that if a new agent joins a group, the outcome can only change in the direction that benefits the new agent.

Notice that many functions defined on single-peaked preferences satisfy participation: selections from the median of the peaks, the mean of the peaks, the minimum or the maximum peak, etc...

**Proposition 1.** Let  $f : \bigcup_{S \in \mathfrak{S}} \mathcal{P}^S \rightarrow \mathbb{R}$  be a function which satisfies participation and is efficient. Then, for all  $S \in \mathfrak{S}$ ,  $P_S \in \mathcal{P}^S$  and  $k \in \mathbb{N}$ , any function  $\varphi$  which selects  $\varphi(P_S, k)$  from the set  $\{d = (x_h, S_h)_{h=1}^k \in D(S, k) \mid d \text{ is efficient and } x_h \in f(P_{S_h}) \text{ for all } h \text{ s.t. } S_h \neq \emptyset\}$  and such that  $\varphi(P_S, 1) = f(P_S)$  is a self-selection consistent choice rule.

Given the construction, all we have to show is that the functions described are well defined, which is equivalent to prove that the set  $\{d = (x_h, S_h)_{h=1}^k \in D(S, k) \mid d \text{ is efficient and } x_h \in f(P_{S_h}) \text{ for all } h \text{ s.t. } S_h \neq \emptyset\}$  is not empty for all  $S \in \mathfrak{S}$ ,  $P_S \in \mathcal{P}^S$  and  $k \in \mathbb{N}$ . We will prove that constructively, by describing a procedure which always converges and finds decisions which belong to the desirable set.

The following Lemmas are useful to show that our construction works.

**Lemma 1.** Let  $f : \bigcup_{S \in \mathfrak{S}} \mathcal{P}^S \rightarrow \mathbb{R}$  satisfy participation. Given  $S, S' \in \mathfrak{S}$  with  $S \cap S' = \emptyset$ , let  $P_S \in \mathcal{P}^S$ ,  $P_{S'} \in \mathcal{P}^{S'}$  and  $y \in \mathbb{R}$  be such that for all  $i \in S'$ ,  $f(P_S) \leq p(i) < y$ , and  $f(P_S) \succ_i y$ . Then  $f(P_S, P_{S'}) \succ_i y$  for all  $i \in S'$ . Furthermore,  $f(P_S) \leq f(P_S, P_{S'}) < y$ .

In words, Lemma 1 states that if a group of agents  $S'$ , whose peaks are to the right of the location chosen by  $f$  for the group  $S$ , joins the group  $S$ , then the choice of  $f$  for the larger group will be to the right of the choice of  $f$  for  $S$ .

**Proof.** Let  $x = f(P_S)$ . Since for all  $i \in S'$ ,  $x \leq p(i) < y$ , and  $x \succ_i y$ , any location in  $[x, y)$  will be preferred to  $y$  by any agent in  $S'$ . Let us see that  $f(P_S, P_{S'}) \in [x, y)$ . For each agent  $i \in S'$ , let  $x_i \in \mathbb{R}$  be such that agent  $i$  is indifferent between  $x$  and  $x_i$ . Notice that for all  $i \in S'$ ,  $x_i < y$ . Then, let us order the agents in  $S'$  by increasing order of their  $x_i$ , and suppose, without loss of generality, that  $S' = \{1, 2, \dots, m\}$ . Take the first agent, and consider the set of agents  $S \cup \{1\}$ . Since  $f$  satisfies participation,  $f(P_S, P_{\{1\}}) \in [x, x_1]$ . Let  $z_1 = f(P_S, P_{\{1\}})$ . Notice that  $z_1 \in [x, x_2]$ . Let  $z_2 = f(P_S, P_{\{1,2\}})$ . Because of participation  $z_2 \succ_2 z_1$ , and therefore  $z_2 \in [x, x_2]$ . If we keep adding agents of  $S'$  to  $S$  in this order, we will finally get that  $z_m = f(P_S, P_{S'}) \in [x, x_m]$ . ■

**Lemma 2.** Let  $f : \bigcup_{S \in \mathfrak{S}} \mathcal{P}^S \rightarrow \mathbb{R}$  be efficient and satisfy participation. Given  $S \in \mathfrak{S}$ ,  $P_S \in \mathcal{P}^S$ , and  $y \in \mathbb{R}$ ,  $y < f(P_S)$ , let  $S' \subset S$  be such that for all  $i \in S'$   $y \succ_i f(P_S)$ . Then,  $f(P_S) \leq f(P_{S \setminus S'})$ .

In a similar vein as Lemma 1, Lemma 2 now states that, if a subset  $S'$  of  $S$  is removed from  $S$ , and all agents in  $S'$  had their peaks to the left of  $f(P_S)$ , then  $f(P_{S \setminus S'})$  will move to the right of  $f(P_S)$ .

**Proof.** Let  $x = f(P_S)$ . For each  $i \in S'$ , let  $x_i \in \mathbb{R}$  be such that agent  $i$  is indifferent between  $x$  and  $x_i$ . Notice that for all  $i \in S'$ ,  $x_i \leq y < x$ . Let us order the agents in  $S'$  by increasing order of  $x_i$ , and suppose, without loss of generality, that  $S' = \{1, 2, \dots, m\}$ . Notice that  $[x_m, x] \subseteq [x_{m-1}, x] \subseteq \dots \subseteq [x_1, x]$ . Let  $z_1 = f(P_{S \setminus \{1\}})$ . By participation,  $x \succ_1 z_1$ , which implies that  $z_1 \notin (x_1, x)$ . By efficiency,  $z_1 \geq x$ . Notice that all the agents in  $S' \setminus \{1\}$  prefer  $y$  to  $z_1$ . Let  $z_2 = f(P_{S \setminus \{1,2\}})$ . By participation,  $z_1 \succ_2 z_2$ , which implies that  $z_2 \notin (x_2, x)$ . Then, by efficiency,  $z_2 \geq x$ . Following this process, we will finally get that, because of participation, agent  $m$  prefers  $z_{m-1}$  to  $z_m = f(P_{S \setminus S'})$ , which implies that  $z_m \notin (x_m, x)$ , and this in turn implies that  $z_m \geq x$  by efficiency. ■

We can now complete the proof of Proposition 1.

**Proof of Proposition 1.** Let  $S \in \mathfrak{S}$ ,  $P_S \in \mathcal{P}^S$  and  $k \in \mathbb{N}$ . If there are at most  $k$  different peaks, we are done. Hence, suppose that there are at least  $k$  different peaks. Let us order the agents by increasing order of their peaks. Let  $S_h^1 = \{i \in S \mid p(i) = p(h)\}$  for all  $h \in \{1, \dots, k-1\}$ , and  $S_k^1 = \{i \in S \mid p(i) > p(k-1)\}$ . Let  $x^1 = (x_h^1)_{h=1}^k$  be such that  $x_h^1 = f(P_{S_h^1})$ , for  $h \in \{1, \dots, k\}$ . Notice that, since  $f$  is efficient,  $x_h^1 = p(h)$  for all  $h \in \{1, \dots, k-1\}$ . Since  $f$  is efficient and all the locations are different, if the decision  $(x_h^1, S_h^1)_{h=1}^k$  is envy-free then it is efficient and we are done. Otherwise, let  $S_{k-1} = S_{k-1}^1 \cup \{i \in S_k^1 \mid x_{k-1}^1 \succ_i x_k^1\}$ . By Lemma 1,  $x_{k-1}^1 \leq f(P_{S_{k-1}}) < x_k^1$ . Notice that we can have agents in  $S_k^1$  which prefer  $x_k^1$  to  $x_{k-1}^1$ , but they have their peaks between  $x_{k-1}^1$  and  $f(P_{S_{k-1}})$ , which implies that they prefer  $f(P_{S_{k-1}})$  to  $x_k^1$ . Let  $S'_k \subset S_k^1$  be the set of those agents. For each  $i \in S'_k$ , let  $z_i$  be such that agent  $i$  is indifferent between  $x_k^1$  and  $z_i$ . Let us order the agents in  $S'_k$  by increasing order of  $z_i$ . Then  $[z_{i+1}, x_k^1] \subseteq [z_i, x_k^1]$ . Take the first agent in this set. By participation  $f(P_{S_{k-1} \cup \{1\}}) \succ_1 f(P_{S_{k-1}})$ , and since  $f(P_{S_{k-1}}) \succ_1 x_k^1$ ,  $f(P_{S_{k-1} \cup \{1\}}) \in [z_1, x_k^1]$ . If  $f(P_{S_{k-1} \cup \{1\}}) \in [z_2, x_k^1]$ , add agent 2 to  $S_{k-1} \cup \{1\}$ , and again by participation  $f(P_{S_{k-1} \cup \{1,2\}}) \succ_2 f(P_{S_{k-1} \cup \{1\}})$ , which will imply that  $f(P_{S_{k-1} \cup \{1,2\}}) \in [z_2, x_k^1]$ . We keep adding agents from  $S'_k$  in the above defined order whenever  $f(P_{S_{k-1} \cup \{1,2,\dots,i\}}) \in [z_{i+1}, x_k^1]$ . Let  $S_k'^1$  be this subset of agents. Then, for all  $i \in S'_k \setminus S_k'^1$ ,  $f(P_{S_{k-1} \cup S_k'^1}) \notin [z_i, x_k^1]$ . Notice that all agents in  $S'_k \setminus S_k'^1$  have their peaks between  $f(P_{S_{k-1} \cup S_k'^1})$  and  $x_k^1$ . Once this process is completed, consider the following sets of agents:  $S_{k-1}^2 = S_{k-1} \cup S_k'^1$ ,  $S_k^2 = S_k^1 \setminus \{\{i \in S_k^1 \mid x_{k-1}^1 \succ_i x_k^1\} \cup S_k'^1\}$ , and  $S_h^2 = S_h^1$  for all  $h \in \{1, \dots, k-2\}$ . Let  $x_h^2 = f(P_{S_h^2})$  for all  $h \in \{1, \dots, k\}$ . We have shown that  $x_{k-1}^2 \geq x_{k-1}^1$ , and by Lemma 2,  $x_k^2 \geq x_k^1$ . If the decision  $(x_h^2, S_h^2)_{h=1}^k$  is envy-free, we are done. If it is not we repeat the process. Notice that, if in step  $j$  we do not get an envy-free

decision, it is because some agents from  $S_h^j$  prefer what agents in  $S_{h-1}^j$  have gotten. Thus, if this is the case, in each step we add agents from  $S_h^j$  to  $S_{h-1}^j$  in the way as described above. The process will end in a finite number of steps because there are a finite number of agents, because by Lemma 1 and Lemma 2 at each step  $x_h^j \geq x_h^{j-1}$  for all  $h \in \{1, \dots, k\}$  and furthermore,  $S_1^{j-1} \subseteq S_1^j$  and  $S_k^j \subseteq S_k^{j-1}$ . ■

We have seen how any function  $f : \bigcup_{S \in \mathfrak{S}} \mathcal{P}^S \rightarrow \mathbb{R}$  satisfying efficiency and participation can be used to induce a self-selection consistent choice rule  $\varphi$ . We now turn to the opposite question. Given a rule  $\varphi$ , its restriction to  $k = 1$  is indeed a function of the above type. What can we say about its properties, if we know that  $\varphi$  is a self-selection consistent choice rule? We can not give a complete answer to this question, but we can provide a definite answer if, in addition,  $\varphi$  satisfies two conditions (continuity and peaks only) which actually only refer to  $\varphi(\cdot, 1)$ .

We first express these two conditions and then state the result.

**Definition 9.** A function  $f : \bigcup_{S \in \mathfrak{S}} \mathcal{P}^S \rightarrow \mathbb{R}$  is peaks-only if for all  $S \in \mathfrak{S}$ , and  $P_S, P'_S \in \mathcal{P}^S$  such that  $p(i) = p'(i)$  for all  $i \in S$ , then  $f(P_S) = f(P'_S)$ .

**Definition 10.** A function  $f : \bigcup_{S \in \mathfrak{S}} \mathcal{P}^S \rightarrow \mathbb{R}$  satisfies continuity if given  $S, S' \in \mathfrak{S}$ ,  $P_S \in \mathcal{P}^S$ ,  $P_{S'} \in \mathcal{P}^{S'}$  such that  $f(P_S) = x$ , and  $f(P_{S'}) = y$ , there is  $n \in \mathbb{N}$  such that  $f(P_S, \dots, P_S, P_{S'}, \dots, P_{S'}) = x$ .

Peaks-only functions are simple in that they only operate on information about each of the agents best alternative. Continuity means that if  $x$  is chosen by a group  $S$  and  $y$  by a group  $S'$ , we can replicate the group  $S$  sufficiently many times so that the union of all these replicas together with  $S'$  would choose  $x$ .<sup>2</sup>

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<sup>2</sup>This definition is borrowed from the social choice literature. See Moulin [8]

**Proposition 2.** *Let  $\varphi : \bigcup_{S \in \mathfrak{S}} \mathcal{P}^S \times \mathbb{N} \rightarrow D$  be a self-selection consistent choice rule such that  $\varphi(\cdot, 1)$  is peaks-only and satisfies continuity. Then,  $\varphi(\cdot, 1)$  satisfies participation.*

**Proof.** Suppose that  $\varphi(\cdot, 1)$  does not satisfy participation. Then there is a group of agents  $S \in \mathfrak{S}$ , an agent  $i \notin S$ , and a preference profile  $(P_S, P_{\{i\}}) \in \mathcal{P}^{S \cup \{i\}}$  such that  $\varphi(P_S, 1) \succ_i \varphi((P_S, P_{\{i\}}), 1)$ . Let us order the agents in  $S \cup \{i\}$  by increasing order of their peaks. Suppose, without loss of generality that  $S \cup \{i\} = \{1, 2, \dots, m\}$  and  $\varphi(P_S, 1) < \varphi((P_S, P_{\{i\}}), 1)$ . Let  $x \in \mathbb{R}$  be such that  $x < \varphi(P_S, 1)$  and every agent in  $S$  prefers anything in the interval  $[p(1), p(m)]$  to  $x$ . Let  $P'_{\{i\}} \in \mathcal{P}^{\{i\}}$  be such that  $p'(i) = p(i)$ , and  $\varphi(P_S, 1) \succ'_i x \succ'_i \varphi((P_S, P'_{\{i\}}), 1)$ . Since  $\varphi(\cdot, 1)$  is peaks-only,  $\varphi((P_S, P'_{\{i\}}), 1) = \varphi((P_S, P_{\{i\}}), 1)$ . Consider an agent  $j$  different from  $i$  and not in  $S$  with  $P_{\{j\}} \in \mathcal{P}^{\{j\}}$  such that  $p(j) = x$ . By efficiency,  $\varphi(P_{\{j\}}, 1) = x$ . Let us consider as many replicas of  $j$  as necessary, say  $n$ , to guarantee that for any possible subgroup  $S' \subseteq S \cup \{i\}$ ,  $\varphi((P_{S'}, P_{\{j\}}, \dots, P_{\{j\}}), 1) = x$ . By continuity this is always possible. Let  $((x_1, S_1), (x_2, S_2)) = \varphi((P_S, P'_{\{i\}}, P_{\{j\}}, \dots, P_{\{j\}}), 2)$ . By efficiency  $x_1 \neq x_2$ . Let us see that all the replicas of  $j$  are in the same group. Suppose this is not the case, then, by no-envy  $x_1$  should be indifferent to  $x_2$  for all agents identical to agent  $j$ . Since  $x_1 \neq x_2$ ,  $x_1 < x_2$ , or  $x_2 < x_1$ . Suppose that  $x_1 < x_2$ , and therefore,  $x_1 < x < x_2$ . By allocation consistency,  $x_1 = \varphi(P_{S_1}, 1)$ . But the preference profile is constructed in such a way that for all agent  $k \in S_1$ ,  $p(k) \geq x$ , and so, by efficiency  $\varphi(P_{S_1}, 1) \in [x, \max_{k \in S_1} p(k)]$ , which contradicts the assumption that  $x_1 < x$ . Consequently, all replicas of  $j$  should be in the same group, say  $S_1$ , and since  $\varphi$  satisfies allocation consistency,  $x_1 = x$ . Let us now see that in  $S_1$  there is no agent except for the replicas of agent  $j$ . By efficiency,  $\varphi(P_{S_2}, 1) \in [p(1), p(m)]$ . But any point in this interval is

preferred for all agents in  $S$  to  $x$ . Thus, by no-envy,  $S_1 \cap S = \emptyset$ . Suppose that  $i \in S_1$ , then  $\varphi(P_{S_2}, 1) = \varphi(P_S, 1)$ , but then agent  $i$  will envy the agents in  $S$ . Therefore  $S_2 = S \cup \{i\}$ , and  $S_1$  is just the set of replicas of  $j$ . By allocation consistency,  $x_2 = \varphi((P_S, P'_{\{i\}}), 1)$ , but then the decision  $((x_1, S_1), (x_2, S_2))$  is not envy-free, because agent  $i$  prefers  $x$  to  $x_2$ , which contradicts the assumption that  $\varphi$  is efficient. Therefore  $\varphi(\cdot, 1)$  satisfies participation. ■

We have already remarked that any selection from the median satisfies Participation, notice that, furthermore, such selections also satisfy peaks-only and continuity. However, neither the mean of the peaks nor the maximum or the minimum peak which satisfy Participation and peaks-only can meet the third requirement of continuity.

#### 4. Self-Selection Consistent rules and the Condorcet Principle.

We now want to investigate whether it is possible to design self selection consistent collective choice functions which respect the Condorcet criterion. For a given preference profile, each set  $S$  of agents and each given  $k$ , an  $S/k$ -decision is a Condorcet winner if no other  $S/k$ -decision would be preferred to it by a majority of agents. The Condorcet criterion requires that, whenever Condorcet winners exist in the set of  $S/k$ -decisions one of them should be chosen. Because our rules must be defined for all profiles, all  $S$  and all  $k$ , the Condorcet criterion imposes a multiplicity of requirements at each profile. We provide a negative result, establishing that no self-selection consistent collective choice function can satisfy the Condorcet criterion (for all  $S$  and  $k$ ). Our negative result can be somewhat qualified in two directions. One by observing that it is not hard to find self selection consistent rules which respect the Condorcet criterion for all  $S$ , when  $k = 1$ . Another qualification is that the negative result comes from our

strict requirement that our collective choice functions should be functions. In a companion paper, we study the added flexibility to be gained by considering multivalued social choice rules.

We now present the formal definitions and results.

**Definition 11.** An  $S/k$ -decision  $d \in D(S, k)$  is a Condorcet winner for  $P_S$  and  $k$  if

$$|\{i \in S \mid d \succ_i d'\}| \geq |\{i \in S \mid d' \succ_i d\}| \text{ for all } d' \in D(S, k)$$

Given  $S \in \mathfrak{S}$ ,  $P_S \in \mathcal{P}^S$  and  $k \in \mathbb{N}$ , let  $CW(P_S, k)$  be the set of  $S/k$ -decisions that are Condorcet winners for  $P_S$  and  $k$ .

Notice that any  $S/k$ -decision that is a Condorcet winner for  $P_S$  and  $k$  is an efficient decision.

**Definition 12.** A collective choice function  $\varphi$  respects the Condorcet criterion if for all  $S \in \mathfrak{S}$ ,  $P_S \in \mathcal{P}^S$  and  $k \in \mathbb{N}$  such that  $CW(P_S, k) \neq \emptyset$ ,  $\varphi(P_S, k) \in CW(P_S, k)$ .

We first show that, whenever a Condorcet winner exists for  $k > 1$ , each of its components is a Condorcet winner for its corresponding group.

**Proposition 3.** Given  $S \in \mathfrak{S}$ ,  $P_S \in \mathcal{P}^S$  and  $k \in \mathbb{N}$ , if an  $S/k$ -decision  $d = ((x_h, S_h))_{h=1}^k \in CW(P_S, k)$ , then  $d_h = (x_h, S_h) \in CW(P_{S_h}, 1)$  for all  $h$  such that  $S_h \neq \emptyset$ .

**Proof.** Suppose that there is an  $h$  such that  $d_h = (x_h, S_h)$  is not a Condorcet winner for  $P_{S_h}$  and 1. Then there is a  $d'_h = (x'_h, S_h)$ , such that

$$|\{i \in S_h \mid d'_h \succ_i d_h\}| > |\{i \in S_h \mid d_h \succ_i d'_h\}|$$

Let  $S_{h1} = \{i \in S_h \mid d'_h \succ_i d_h\}$ ,  $S_{h2} = \{i \in S_h \mid d_h \succ_i d'_h\}$ , and  $d' = (d'_h, d_{-h})$ . Then,  $\{i \in S \mid d' \succ_i d\} = S_{h1}$ , and  $\{i \in S \mid d \succ_i d'\} = S_{h2}$ . But then,  $|\{i \in S \mid d' \succ_i d\}| > |\{i \in S \mid d \succ_i d'\}|$ , which contradicts the assumption that  $d$  is a Condorcet winner for  $P_S$  and  $k$ . ■

Rules satisfying the condition of Proposition 3 would partially respect the Condorcet criterion. The reader can check that it is not hard to find self-selection consistent rules which respect the Condorcet principle to that extend. Unfortunately, our last result shows that a complete respect of the principle will be incompatible with self selection consistency.

**Proposition 4.** *No self-selection consistent rule respects the Condorcet criterion.*

**Proof.** It will be sufficient to prove that a violation of the Condorcet criterion must arise for any rule at some specific profiles and some  $S$  and  $k$ . Let  $S = \{1, 2, \dots, 7, 8\}$ , and  $P_S = (\succ_i)_{i=1}^8$  be such that for all  $i$ ,  $\succ_i$  is euclidean on  $\mathbb{R}$  with the following peaks:  $p(1) = 2$ ,  $p(2) = 6$ ,  $p(3) = 8$ ,  $p(4) = 11$ ,  $p(5) = 15$ ,  $p(6) = 18$ ,  $p(7) = 20$ ,  $p(8) = 24$ . Consider first the set of agents  $S' = \{1, 2, 3, 4, 5\}$ . First, notice that no efficient  $S'/2$ -decision  $d = (x_h, S'_h)_{h=1}^2$ , such that  $2 \leq x_1 < 6$  and  $11 \leq x_2 \leq 15$  can be a Condorcet winner because if  $x_2 < 15$ , the decision  $((6, \{1, 2, 3\}), (15, \{4, 5\}))$  defeats  $d$ , and if  $x_2 = 15$ , then the decision  $((6, \{1, 2, 3\}), (11, \{4, 5\}))$  defeats  $d$ . Second, notice that no efficient  $S'/2$ -decision  $d = (x_h, S'_h)_{h=1}^2$ , such that  $x_1 = 6$  and  $12 < x_2 \leq 15$  can be a Condorcet winner because for a sufficiently small  $\varepsilon > 0$ , the decision  $((2, \{1, 2\}), (10 - \varepsilon, \{3, 4, 5\}))$  defeats  $d$ . Because of Proposition 3 and given that none of the decisions described above can be Condorcet winners, the set of possible candidates is reduced to the set of decisions  $d = (x_h, S'_h)_{h=1}^2$  such that  $x_1 = 6$ ,  $S'_1 = \{1, 2, 3\}$  and  $11 \leq x_2 \leq 12$ ,  $S'_2 = \{4, 5\}$ . In particular notice

that the decision  $((6, \{1, 2, 3\}), (11, \{4, 5\}))$  is one, but not the only among the Condorcet winners in our set. Now, a self-selection consistent rule which respects the Condorcet criterion should select a Condorcet winner among those identified above, and any such choice should be such that  $11 \leq \varphi(P_{S'_2}, 1) \leq 12$ . In a completely symmetric way, consider now the set of agents  $S'' = \{4, 5, 6, 7, 8\}$ . The set of  $S''/2$ -decisions which are candidates to Condorcet winners is the set of decisions  $d = (x_h, S'_h)_{h=1}^2$  such that  $14 \leq x_1 \leq 15$ ,  $S'_1 = \{4, 5\}$  and  $x_2 = 20$ ,  $S''_2 = \{6, 7, 8\}$ . Thus, a self-selection consistent rule which respects the Condorcet criterion should select a Condorcet winner from that set, and it should be such that  $14 \leq \varphi(P_{S''_1}, 1) \leq 15$ . But, the set  $S''_1$  is exactly the set  $S'_2$ . Therefore  $\varphi(P_{S'_2}, 1)$  should be equal to  $\varphi(P_{S''_1}, 1)$  which is incompatible with the restriction that  $\varphi(P_{S'_2}, 1) \in [11, 12]$  and  $\varphi(P_{S''_1}, 1) \in [14, 15]$ . Consequently, there is no self-selection consistent rule which respects the Condorcet criterion<sup>3</sup>. ■

## 5. Conclusion.

When designing and analyzing social choice rules, it is important to understand how the choices made by these rules would change in accordance to changes in the parameters defining the characteristics of society. What parameter changes are deemed relevant will depend on the type of decisions to be made. For example, when social choice rules are aimed at solving distribution problems, then a very relevant parameter is the total amount of resources to be distributed. Then, for any given rule, one may study how the shares of different agents in society would change in response to changes in the size of the pie. A rule could then be called allocation consistent if its response to changes in the size parameter is adequate

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<sup>3</sup>Although we do not have a formal proof, we conjecture that 8 is the minimal number of agents that we need in order to get the negative result.

in some sense. Of course, different notions of consistency will arise, depending on what parameter changes we concentrate upon, and of what we think is an “adequate” change in parameters. In this paper we have concentrated in a consistency requirement that is based in comparing the choices for societies of different size making decisions which involve different degrees of complexity. Our concept of consistency is innovative in that it only applies to changes in the composition of society which are determined endogenously by the very structure of the social decisions. Ours is a concept of internal consistency between the decisions of society at large and the decisions that some segments of society would make if they could concentrate on their partial concerns. It is in contrast with most notions of consistency, which we could call external, since they establish requirements on the consequences of any sort of exogenous change in parameters. While our major concern was conceptual, we have reached rather sharp conclusions on the possibility that social choice rules might satisfy our notion of self selection consistence. We’d like to conclude by stressing three points that were already made within the text, regarding our modelling choices and the possibility of extending our present analysis. First of all, notice that our results apply to a simple world where decisions can be represented as  $k$ -tuples of points in a line and agents’ preferences are single peaked. While this is the natural extension of a most popular model, it would be interesting to explore the consequences of more general formulations. Second, as already remarked in Section 4, some of our results depend on the fact that we study social choice functions (allowing for a single choice by each group of agents and decision size), rather than social choice correspondences. A final and deeper remark refers to a basic modelling choice which we already stressed in Section 4. For simplicity, we have made the strong assumption that agents are interested in one and only one of the multiple simultaneous decisions taken

by society. This leads to a partition of agents into disjoint interest groups. A more realistic formulation should allow for agents who are concerned with more than one aspect of society's global decision (maybe in varying degrees). This would lead to more complex subdivisions of societies into partially overlapping groups. It would be interesting to extend the notion of self-selection consistency to analyze the internal cohesion induced by different social choice rules in these more complex and realistic setups.

## References

- [1] Alesina, A., and E. Spolaore, On the Number and Size of Nations, *Quart.J.Econ.* (1997), 1027-56.
- [2] Barberà, S., and C. Beviá, Self-Selection Consistency and the Condorcet Criterion, mimeo (2000), Universitat Autònoma de Barcelona.
- [3] Greenberg, J., and S. Weber, Stable Coalition Structures with a Unidimensional Set of Alternatives, *J.Econ.Theory* 60 (1993), 62-82.
- [4] Jehiel, P., and S. Schotchmer, Free Mobility and the Optimal Number of Jurisdictions, *Ann.Econ.Statist.* 45 (1997), 219-31.
- [5] Konishi, H., M. LeBreton, and S. Weber, Equilibrium in a Finite Local Public Goods Economy, *J.Econ.Theory* 79 (1998), 224-44.
- [6] Milchtaich, I., and E. Winter, Stability and Segregation in Group Formation, mimeo (1998), Hebrew University of Jerusalem.
- [7] Miyagawa, E , “Mechanisms for Providing a Menu of Public Goods”, Ph.D. Thesis, University of Rochester, 1998.
- [8] Moulin, H, “Axioms of Cooperative Decision Making”, Cambridge University Press, 1988.
- [9] Thomson, W, “Consistency and Its Converse: An Introduction”, Rochester Center for Economic Research. Working Paper No.448, 1998.
- [10] Tiebout, C , A Pure Theory of Local Public Expenditure, *J.Polit.Economy* 64 (1956), 416-424.