Meritocracy, Egalitarianism and the Stability of Majoritarian Organizations*

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Abstract

Egalitarianism and meritocracy are competing principles to distribute the joint benefitts of cooperation. We examine the consequences of letting members of society vote between those two principles, in a context where groups of a certain size must be formed in order for individuals to become productive. Our setup induces a hedonic game of coalition formation. We study the existence of core stable partitions (organizational structures) of this game. We show that the inability of voters to commit to one distributional rule or another is a potential source of instability. But we also prove that, when stable organizational structures exist, they may be rich in form, and different than those predicted by alternative models of group formation. Non-segregated groups may arise within core stable structures. Stability is also compatible with the coexistence of meritocratic and egalitarian groups. These phenomena are robust, and persist under alternative variants of our initial model.

Key words: Egalitarianism, Meritocracy, Coalition Formation, Hedonic Games, Core Stability, Assortative Mating.

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1. Introduction

Egalitarianism and meritocracy are two competing principles to distribute the joint benefits from cooperation. One could debate their relative merits and side for one or the other. Rather, we analyze the consequences of not taking sides between these two principles, and letting different organizations choose by vote between the two, in a context where this choice is part of group formation decisions. The lack of ability to commit "a priori" on one specific distributional criterion may lead to organizational structures and consequences that would not arise in other frameworks. Meritocracy and egalitarianism may coexist within stable societies. Moreover, stability is compatible with the formation of groups where diverse individuals seek to cooperate with each other, rather than always preferring to work with their equals. Assortative mating or homophily are not the only rule in our world, where distributional considerations may give rise for an interest in diversity, and non-segregation is compatible with stability.

These important structural facts are proven to arise in natural, non-pathological circumstances, from a simple stylized model. Specifically, we consider societies composed of n individuals within which groups must form to perform a certain task. Each individual is endowed with a productivity level. Groups need a minimal size, v, to be productive. Beyond that size, group production is the sum of the productivities of its members. Agents prefer to get a higher than a lower pay. If they must choose among organizations that will pay them the same, they prefer those whose average productivity is higher. If a group is formed, its members decide by majority vote whether to distribute their product according to meritocracy or to egalitarianism. Hence, the median voter in each group ends up determining the distributional rule: it will be meritocratic if the median's productivity is above the group's mean, egalitarian otherwise.

We examine the consequences of this form of group governance on the size, stability and composition of organizations, on their endogenous choice of rewards, on their ability to compete for talent, and their ability to keep a competitive edge under changes in their definitional parameters.

Under the assumption that agents know the productivities of all others, they can anticipate what rewards they will get from joining any given group. They will thus play a hedonic game (Drèze and Greenberg (1980)), where agents have preferences over the groups they may belong to and outcomes are partitions of agents into groups.

Our highly stylized model allows for different interpretations. In all its apparent simplicity, it allows us to touch upon a variety of topics that are at the forefront of today's economic research.

What we offer is a very compact view of the forces that may drive the different members of the same society to stick to one group and to dissociate from others.

An interpretation of our model approaches it to the work on country formation and secessions (Alesina and Spolaore (1997), Le Breton et al. (2004)), except that we abstract from geography and instead highlight the role of distributional issues as a driving force that shapes different types of societies. Stability issues will be central in our case, as they are in that related literature: one of our goals is to characterize core stable country configurations (organizational structures). Under this country formation interpretation, our model is suggestive of different phenomena that have been recently highlighted in the literature, regarding the differences in characteristics among advanced societies. We prove that different countries may adopt different distributional criteria and still coexist, as in the literature on "varieties of capitalism" (Acemoglu et al (2012), Hall and Soskice (2001)). Of course, our static analysis cannot fully encompass all the dynamic and incentive aspects of a more complex setup, but it is significant that this important stylized fact arises from such a simple model as ours. Moreover, we can provide a highly suggestive comparative static analysis pointing at incentive issues. We present examples where changes in the population and/or in the threshold size required for organizations to become productive have consequences on the country's ability to compete for highly qualified individuals.

Other interpretations of our model reflect on the formation of organizations or jurisdictions within a community. Leading interpretations include the establishment of decentralized regions, public university systems, cooperative firms or partnerships within regulated professions. In that respect, our contribution is related to the literature on the endogenous formation of institutions in broad terms (Caplin and Nalebuff (1994)), or the more specific ones analyzing the choice of regional tax systems, local public goods, clubs and sorting (Tiebout (1959), Schelling (1969), Farrell and Scotchmer (1988), Epple and Romer (1991), Ellickson et al (1999), Hansen and Kessler (2001), Puy (2007), Damiano et al (2010), among others). However, our results are different, both on technical grounds and also regarding some basic conclusions. For example, they are in contrast with the usual conclusion that stability pushes agents to form segregated groups. Our model is one where segregation may or may not arise, and the structural characteristics of the distributions leading to non-segregation can be clearly traced to the fundamentals of the productivity distribution.

The analysis of hedonic games is never a trivial task. It always depends on the type of preferences over groups of agents that are admissible in the worlds under consideration. In our case, the family of preferences that agents may have over groups is dictated by the structure of the model and by the role that voting plays over the distribution of the benefits from cooperation. The games we confront are thus more specific than those one could postulate without reference to any particular interpretation (see Banerjee et al (2001), Bogomolnaia and Jackson (2002)), and as a result we are able to obtain clear-cut existence and characterization results for core stable coalitional structures[1]. At the same time, our model differs from others that also give rise to specific hedonic games but restrict the preferences of agents in alternative manners (see for example Alcalde and Romero-Medina (2006), Iehlé (2007), Bogomolnaia et al (2008), Papai (2011), Pycia (2012)). All of them apply to domains of preferences different than those implied by our model. For example, Pycia's includes matching problems as a special case, but then does not apply to our world because we implicitly assume an equal treatment of equals property that is not present in the matching literature. Since we cannot rely on preceding work that derives from different models, we offer a complete treatment of existence and characterization issues as an integral part of our study. The main lesson from that analysis is that all the richness of situations we may obtain, with the possibility of segregated and nonsegregated groups, as well as egalitarian and meritocratic rewards all coexisting at equilibrium, is by no means exceptional, but a real and natural possibility.

Let us now be more specific about our formal results regarding the existence of core stable organizational structures and their characteristics.

We first analyze the case where potential groups to be formed are so large relative to the population that only one at most can be formed. That case is interesting on its own. But we also emphasize it because the existence of stable organizational structures is shown to depend on the satisfaction of a condition, the weak top property, that is sufficient for stability under any general hedonic game, and in that case also turns out to be necessary. The weak top property, first introduced in Banerjee at al (2001), is used at that point but also along the rest of the paper.

Next, we turn to the analysis of three type societies, where individuals are restricted to have three possible productivity levels: high, medium and low. Modeling a society through such a three-way partition is certainly limitative, but also a reference case, that is resorted to in other contexts. People are classified by social status into the elite, the middle and the lower class; countries are classified into developed, developing and less developed, etc.... A major contribution of our paper is the full characterization for this special case, and its generalization to what we call three-way clustered societies, that is, societies with an arbitrary number of types but whose members are

[1] For precise definitions we refer to the next section

clustered into at most three distinct groups of agents whose productivities are "similar" within each cluster and yet "sufficiently differentiated" across them.

The detailed study of three-way clustered societies is complex, but its essential features can be grasped by a close examination of the case where there are only agents with three types and $v = n/2$, that is where the number of agents in society allows for the possibility of just forming two groups. In that case, the conditions under which non-segregated groups necessarily arise as part of the unique core stable organizational structure become transparent. Interestingly, these cases correspond to situations where the predominant type in society is the middle one, while high and low types are relatively few.

In order to probe the robustness of our previous results, we present evidence that the same basic phenomena would arise from alternative specifications than those of our basic model.

We show that our main results persist under a continuous specification of the model. In particular, if a continuous population with a symmetric and single peaked density of productivities must organize in groups that include at least $1/2$ of the total population, the segregated organization is never stable. Instead, the core contains a polarized meritocratic group containing individuals from the top and from the bottom of the productivity distribution, along with a complementary group of individuals from middle range productivities.

Finally, we test the robustness of our results to a specification where individuals contribute effort to their group voluntarily. In this new set up meritocracy is easier to sustain, and this works in favor of stability of segregated organizations. However, non-segregated organizations, as well as egalitarian groups are still compatible with stability.

The paper is organized as follows. After this introduction, Section 2 presents the basic model and discusses a variety of examples that announce the main messages of the paper. Sections 3, 4 and 5 discuss the existence of stable organizational structures, and their characteristics under different situations. In Section 3 we emphasize sufficient conditions and their consequences for the case where productive organizations must be large relative to society's size. Section 4 is devoted to three type societies, and Section 5 extends its results to the case of three clustered societies. Sections 6 and 7 discuss alternative modelling choices. In 6 we study a model with a continuum of agents, and in 7 we introduce the possibility of variable effort levels. Section 8 concludes.

2. The basic model and its derived organizational structures

In this section we present our basic model, and we then illustrate, by means of examples, the richness of implications that arise from it, regarding the variety of organizational structures that may occur in stable societies, their sensitivity to different distributions of productivities, and the role of voting as a possible source of instability. The examples are also used to provide an overview of some of the formal results that will be presented in subsequent sections, on the existence and characterization of stable organizational structures.

Let us remark that in addition to the basic model we are about to present, we shall also introduce several variants of it in Sections 6 and 7, in order to prove that the phenomena we describe are robust to alternative specifications.

Let $N = \{1, 2, ..., n\}$ be a set of n individuals characterized by their individual potential productivities $\lambda = (\lambda_1, ..., \lambda_n)$ with $\lambda_1 \geq ... \geq \lambda_n \geq 0$. Subsets of N are called *groups* or *organizations* interchangeably. Individuals can only become productive if they work within a group $G \subseteq N$ of size at least v. Groups of smaller size produce nothing, while groups of size v or larger produce the sum of their members' productivities. A *society* is represented by a triple (N, λ, v).

We refer to a group of cardinality less than v as being *unproductive.* The top set $T = \{1, ..., v\}$ contains the first v agents in terms of productivity. Similarly for any $G \subset N$, $T(G)$ denotes the first v agents in G.

We denote the average productivity of a group $G \subseteq N$ by $\bar{\lambda}_G$, and by λ_G is the vector of productivities of the agents in G.

If a productive group is formed, its total production must be distributed among the agents of the group. Agents prefer to get a higher than a lower pay. Lexicographically, if they must choose among organizations that will pay them the same, they prefer those whose average productivity is higher. [2]

Productive groups internally decide, by majority voting, whether to distribute their product in an egalitarian or in a meritocratic manner. That is, whether all agents in the organization G get the same reward, $\bar{\lambda}_G$, or each one is rewarded by its productivity, λ_i. There is no way to commit a priori to any of these two principles. A majority in group G will favor meritocracy if the productivity

[2] We adopt this lexicographic specification of preferences as the simplest way to represent the fact that, in addition to the material reward, individuals may also value other dimensions of the participation in a group, like prestige. Other specifications that reflect richer trade-offs between individual rewards and the "quality" are possible, but make the model less tractable. Most of our results apply to setups with preferences without the lexicographic component (see our concluding remarks).

of the median, $\lambda_{m(G)}$, is greater than $\bar{\lambda}_G$. Otherwise, the majority will be for egalitarianism. Ties are broken in the following way: if there are more than one median agent, ties are broken in favor of the agent with the highest productivity. If the productivity of the median agent is equal to the mean productivity, we consider that the group is meritocratic.

Since agents know the rules and also the productivities of all others, they can anticipate what rewards they will get from joining any given group. They will thus play a hedonic game (Drèze and Greenberg (1980)), where outcomes are partitions of agents into groups. A natural prediction is that stable partitions will arise from playing these games. The following definitions formalize the stability concept that we use in this paper.

Definition 1. *Given a society (N, λ, v), an organizational structure is a partition of N denoted by π. Two organizational structures, π and π', are equivalent if for all $G \in \pi$ there is $G' \in \pi'$ such that $\lambda_G = \lambda_{G'}$ and viceversa. A group G is segregated if given i and j in G with $\lambda_i < \lambda_j$, and $k \in N$ such that $\lambda_i \leq \lambda_k \leq \lambda_j$, $k \in G$. An organizational structure is segregated if all the groups in the partition are segregated.*

Definition 2. *An organizational structure is blocked by a group G if all members in G are strictly better off in G than in the group they are assigned in the organizational structure. An organizational structure is core stable if there are no groups that block it.*

We now present different examples that will illustrate the richness of implications arising from the fact that agents do vote on distributional issues.

Our first example shows two important and independent points. The first one is that in a core stable organizational structure different reward systems may coexist. The second one is that a core stable organizational structure may be non-segregated. In the interpretation where groups are different institutions that form within a country, the example shows how people that are diverse may find an advantage to get together for distributional reasons.

Example 1. *A society with stable non-segregated organizations where different reward systems coexist.*

Let $N = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $v = 5$, and $\lambda = (100, 100, 75, 75, 75, 75, 75, 75, 75, 45)$. Let $G_1 = \{1, 3, 4, 5, 10\}$ and $G_2 = \{2, 6, 7, 8, 9\}$. Note that G_1 is meritocratic and G_2 is egalitarian. Let us see that the organizational structure $\pi = (G_1, G_2)$ is core stable. Note that the medium type agents in G_2 can only improve if a high type is added to the group or if a medium type is substituted by a *high type. But since the other high type not in G_2 is already in a meritocratic group, he does not have incentives to form the potential blocking group. The two high types cannot be together in a meritocratic group, and any other agent needs high types to improve. That implies that π is a core stable organizational structure. Note that high and medium productivity agents are split between the two groups. Any other core stable organizational structure is equivalent to this one.*

Notice, for further reference, that this is an example of a society with three types of individuals, defined by three different productivity levels, and whose numbers allow to form exactly two productive groups. We'll see in Section 4 that for societies with these characteristics, stability is always guaranteed (Proposition 4) and that in fact this is a special case where the core stable organizational structures are necessarily non-segregated and unique (Proposition 5).

At this point, it is worth comparing the results from our model with those that one would obtain if there was no choice of distributional rule, or if that choice was open to unrestricted bargaining. If agents were forced to adopt a fixed distributional rule, either meritocracy or egalitarianism, there would always exist a segregated stable organizational structure. That would be the one where the v most productive agents get together, then the next v most productive form a second group, and so on, thus eventually leaving some agents out of any productive group. Segregation would also be the consequence of stability in the polar case where agents could freely bargain how to distribute the gains from cooperation. Indeed, when $n = kv$, for some integer k, the unique stable structure would again be the one we just described, under a meritocratic reward scheme. By contrast, the ability of societies to vote between our two distributional criteria is what gives rise to the possibility of non-segregated stable organizational structures[3].

Under free bargaining instability will be the rule when exactly k groups of size v can not form, because agents left into non-productive groups would have reservation value zero and could be offered low rewards to form blocking coalitions. As we shall see, instability may also be unavoidable in our model for some societies. But, as shown in the examples that follow (Examples 2 and 3), it will arise in subtler ways than in the case of bargaining, while stability will be perfectly compatible with some agents being left into unproductive groups (Example 4).

Our next two examples show that stability is not always possible, and point at reasons that we later will examine in general terms.

[3] In Bogomolnaia et al (2008) non-segregated groups also arise from the combination of voting and group formation. In their model agents in a group decide by vote the location of a public good, but share its cost equally.

Example 2. *A society with no core stable organizational structures.*
Let $N = \{1, 2, 3, 4, 5\}$, $v = 3$, *and* $\lambda = (100, 84, 84, 84, 60)$. *Let us see first that in any core stable organizational structure the high productivity agent cannot belong to a productive meritocratic group. This is because in this example the grand coalition is the highest mean meritocratic group containing the high productivity agent. But the organizational structure composed by the grand coalition is not core stable because the medium type agents on their own can form a meritocratic group with a higher average productivity. Now, let us see that in any core stable organizational structure the high productivity agent cannot belong to a productive egalitarian group. Like before, we only need to check that the organizational structure where the productive group is the egalitarian group with the highest mean productivity is blocked. This productive group is the one formed by the high type agent with two of the medium type agents. This organizational structure is blocked by the meritocratic group formed by the high type agent, the third medium type agent and the low type agent. Finally, note that the high type agent cannot belong to an unproductive group either, because the productive one formed by the high type and two medium types blocks any organizational structure where the productive group does not contain the high type. Thus, there is no core stable organizational structure.*

For further reference, notice that in this example it is only possible to form one productive organization, at most. Thus, we are in a world where the minimal size of productive organizations is large relative to the overall population. This case is examined in Section 3, and there we provide a necessary and sufficient condition for existence of stable organizational structures. This condition requires societies to satisfy the weak top property, a sufficient condition for stability under any general hedonic games that turns out to be also necessary in this case.

In the following example we show that the issue of existence may also arise in societies allowing for larger numbers of potential groups, and even if a priori there is no need to leave any agent outside of a productive group.

Example 3. *A society with no core stable organizational structure,* $n = 3v$.
Let $N = \{1, ..., 9\}$, $\lambda = (100, 75, 75, 75, 25, 25, 25, 25, 25)$, $v = 3$. *In order to prove that no organizational structure is stable, it is enough to show that, in an stable structure, the high type productivity agent cannot belong to an unproductive group, cannot be part of an egalitarian group, and cannot be part of a meritocratic group. Clearly, if the high productivity agent is in unproductive group, no matter how the other agents are organized, medium type agents will always prefer*

to form a productive group with the high type one. If the high type is in an egalitarian group, it has to be the one with the greatest mean that leaves behind one of the medium type agents. The rest of the society has to be organized in a stable way, which implies an egalitarian group with productivities $(75, 25, 25)$ *and a meritocratic group with productivities* $(25, 25, 25)$*. The high type agent, together with the medium type agent in the second group and a low type agent in the third group, can form a meritocratic group that blocks that organization. Finally, if the high type is in a meritocratic group, this group contains medium type agents, but independently of how the rest of agents are organized, the group of medium type agents blocks that organization.*

Once more, for further reference, observe that we are again in a case with only three types of agents, as in Example 1. However, the size of society now allows for more than two productive groups to form, whereas in Example 1 only two groups at most could arise. As we shall see in subsection 4.2, this larger relative size of society does no longer guarantee that stability holds. More specifically, our example here fails to satisfy a condition that we identify in subsection 4.2 and Proposition 5, as being necessary and sufficient for the existence of stable organizational structures in general, three type societies[4].

The next example shows that, unlike in the case where individuals could freely bargain for their rewards, instability is not necessarily associated with the existence of agents who are left out of productive groups. It also shows that even in the event where several groups of minimal productive size could form, stability may generate the emergence of larger groups.

Example 4. *A case where* $n = kv$*, and yet no partition of agents into groups of of size* v *can achieve stability.*
Let $N = \{1, 2, 3, 4, 5, 6\}$, $v = 3$, *and* $\lambda = (50, 40, 40, 35, 25, 10)$*. Let* (P, U) *be an organizational structure where* $P = \{1, 2, 3, 5\}$ *and it is meritocratic and* $U = \{4, 6\}$ *and it is an unproductive group.* (P, U) *cannot be blocked because* P *is the meritocratic group with the highest mean and the only agent that could improve without using anyone from* P *is agent* 4 *but* $\{4, 5, 6\}$ *is meritocratic. The egalitarian group with the greatest mean in* $E = \{1, 2, 3\}$, $N \backslash E$ *is meritocratic. The organization* $(E, N \backslash E)$ *is blocked by* $G = \{1, 4, 6\}$ *which is a meritocratic group with a greatest mean than* $N \backslash E$*. Any organization with two meritocratic groups or one meritocratic and one unproductive group is blocked by* P*, any organization with two egalitarian groups or one egalitarian*

[4] This condition requires that societies be structured, according to Definition 6 in subsection 4.2.

and one unproductive group is blocked by E. It can be checked that any other organization is blocked by P. Thus, (P, U) is the unique core stable organizational structure.

The examples that follow are intended to show that our model is amenable to perform some comparative static analysis. Before we introduce them, let us clearly state that this type of exercise is well grounded, because in the sections that follow we shall identify conditions guaranteeing that core stable equilibria are "almost unique", in a well defined sense. Our examples below conform to society characteristics impliying almost uniqueness, as it was also the case in our Example 1.

We first remark that the issue of stability is related to the size of minimal productive groups in a non-trivial manner.

Example 5. *Changes in v can be either stabilizing or de-stabilizing.*
Let $N = \{1, 2, ..., 7\}$ and $\lambda = (100, 84, 84, 84, 84, 60, 60)$. Suppose that initially $v = 4$.
Note that medium type agents can form a group by their own with a payoff of 84. The egalitarian group with the greatest mean is blocked by the meritocratic group containing the high, one medium and two low type agents. Any meritocratic G with the high type is blocked by the four medium agents together. No organizational structure is stable.
But, if $v = 3$, (G_1, G_2, U), with $G_1 = \{1, 2, 6\}$ and $G_2 = \{3, 4, 5\}$ both meritocratic and $U = \{7\}$ unproductive is a core stable organizational structure, because the high type is in a meritocratic group and he cannot increase the mean above 84 while keeping meritocracy.

Our last example is suggestive of a variety of applications that might derive from our model, if embedded in a more general setting. Let us first exhibit the example and then comment on its possible implications.

Example 6. *Changes in v and n can modify the distributional criteria in stable organizational structures.*
Let $N = \{1, ..., 14\}$, $v = 7$, $\lambda = (10, 10, 7, 7, 7, 1, 1, 1, 1, 1, 1, 1, 1, 1)$.
The organization structure $(T, N\backslash T)$, where T is meritocratic and $N\backslash T$ is egalitarian is core stable. Assume now that the size of groups, and the set of potential participants must be reduced to $v' = 5$ and $N' = \{1, ..., 10\}$. It would seem natural to fire the two worse people of each group, so that the productivities in this new society are $\lambda' = (10, 10, 7, 7, 7, 1, 1, 1, 1, 1)$. The organization $(T', N\backslash T')$, remains core stable for (N', v', λ'). Yet, in that case, the top group in this organization becomes egalitarian. Whereas, if the first organization would have fired two of the medium productivity

agents, rather than the two low ones, the core stable partition of the resulting smaller society would still be meritocratic.

This example has been chosen to identify the potential consequences of parameter changes that may be interpreted as budget cuts. As an illustration, the fourteen agents in the society may stand for the set of individuals who have the credentials to joint a university system, with $v = 7$ being the minimal size that each university must have in order to be allowed to operate. The passage from fourteen to ten agents may represent the government's decision to limit the number of teachers that may be hired. The change of v from 7 to 5 can be seen as a possible relaxation of the universities' accreditation requirements. Then, our example shows that the "obvious" choice of agents to be dismissed may result in a dramatic change in distributional criteria. We do not want to exaggerate the importance of the example, but notice that it could become the starting point of a study regarding the ability of societies to compete in a larger world. What happens in the example is that the best university may end up shifting from meritocracy to egalitarianism at equilibrium. We have not modeled external competition for high level faculty, but we could assume that the most productive agents are likely to get outside options involving rewards higher that average. Under these unmodeled but reasonable assumption, our example is a warning that budget cuts may have a high decapitalizing effect in societies where distributional decisions are made by the majority.

Similar and apparently anomalous phenomena would arise as the potential result of other parametric changes. In the same example, if the low type members would upgrade their qualifications close to the medium type, say from 1 to 6, meritocracy would also be lost in stable organizational structures.

3. Sufficient conditions for core stability, and their necessity when organizations must be "large"

Simple sufficient conditions assuring the existence of core stable organizational structures are easy to describe. For any distribution of productivities guaranteeing that segregated groups are meritocratic, any organization of society into segregated groups of minimal size is core stable. This is the case for example, under a uniform or concave distribution[5], that is, when for any three consecutive agents i, j, k with $\lambda_i \leq \lambda_j \leq \lambda_k$, $\lambda_k - \lambda_j \leq \lambda_j - \lambda_i$. Other environments where the existence of core

[5] In the dual case, where all segregated groups are egalitarian, the organization of society into segregated groups of minimal size does no longer assure core stability.

stable organizational structures is guaranteed are those where all agents have the same productivity (one type societies), or are divided into two sets H of n_H identical individuals of high type and a set L of $n - n_H$ identical individuals of low type (two type societies). Existence in the first case is trivial. In the second case, if $n_H \geq v$, the organizational structure (H, L) is trivially core stable. If $n_H < v$, the reader may check that the organizational structure $(T, N \backslash T)$ is also core stable. In the following section we shall discuss the much richer case where agents come in three different types, and show that existence issues become challenging then.

We now turn attention to a more general condition, that is in fact sufficient for existence of core stable organizational structures in general hedonic games: *the weak top group property* (Banerjee et al, 2001). We begin by proving that identifying weak top groups in our model, when they exist, is an easy task (Proposition 1). In addition to its intrinsic interest, this result is used in subsequent sections, when searching for potential candidates to form core stable organizational structures. We then show that the weak top property has additional bite in societies where $n < 2v$, and thus only one productive group can be formed, at most. For these simple societies requiring large minimal size organizations relative to total population, the weak top group property is necessary and sufficient for core stable organizational structures to exist (Proposition 2). Finally, we identify those societies that are sufficiently small for existence to be guaranteed in any case (Proposition 3).

Definition 3. *A group $W \subseteq G \subseteq N, W \neq \emptyset$, is a weak top group of G if it has an ordered partition $(S_1, ..., S_l)$ such that (i) any agent in S_1 weakly prefers W to any subset of G, and (ii) for any $k > 1$, any agent in S_k needs cooperation of at least one agent in $\cup_{m<k} S_m$ in order to form a strictly better group than W. A game satisfies the weak top group property if for any group $G \subseteq N, G \neq \emptyset$, there exists a weak top group W of G.*

If the weak top group property is satisfied, a core stable organizational structure, $(G_1, ..., G_m)$ always exists and can be constructed by sequentially selecting weak top groups from the population: G_1 is the weak top group of N, G_2 the weak top group of $N \backslash G_1$, and so on. [6]

We can now show that in our model, weak top groups, if they exist, must have a very specific and simple form. This fact will greatly simplify our discussion of stability, and is therefore an important step to be repeatedly used in our proofs.

[6] Stronger conditions can be found in the literature that guarantee core stable organizational structures. For example, the Top Group Property (TGP), requires that any group G of agents contains a subgroup that is the best subset of G for all of its members (Banerjee et al, 2001). The TGP is a relaxation of the common ranking property introduced by Farrell and Scotchmer (1988). Under those conditions the core is nonempty and it has a unique element.

Before discussing the form of weak top groups, let's introduce the notion of a congruent group (Le Breton et al (2008)).

Definition 4. *A group $C \subseteq G \subseteq N, C \neq \emptyset$, is a congruent group of G if for all $i \in C$, and for all $S \subset G$ such S is a strictly better group than C for i, there is an agent $j \in S \cap C$ such that C is a strictly better group than S for this agent j.*

Note that any weak top group of G is a congruent group of G.

We can now state our characterization result for weak top groups of G.

Proposition 1. *Let $M_{+}(G)$ be the set of meritocratic groups of G with the greatest mean, and let $E_{+}(G)$ be the set of egalitarian groups of G with the greatest mean. A group W is weak top group of G if and only if it is a congruent group of G, and either belongs to $M_{+}(G)$ or to $E_{+}(G)$.*

The proof is presented in the Appendix.

Consider next societies where organizations must be relatively large so that only one productive group can be organized, i.e. $v > n/2$. In these societies the weak top group property is necessary and sufficient for the existence of core stable organizational structures.

Proposition 2. *A society where $v > n/2$ has core stable organizational structures if and only if N has a weak top group.*

Proof. Sufficiency is clear: just partition the society into the weak top group of N and leave the other agents together in an unproductive group.

Necessity follows from the fact that if a partition $\pi = (P, N \backslash P)$ is in the core, $P \in M_{+}(N)$ or $P \in E_{+}(N)$. Since π cannot be blocked, there is no group $S \subseteq N$ such that all $i \in S \cap P$ are better off in S than in P. Thus, P is congruent and by Proposition 1 it is a weak top group of N. ■

A direct application of Proposition 2 is the following.

Proposition 3. *In societies where $v > 2n/3$ a weak top group of N always exists. Therefore, there are always core stable organizational structures.*

Proof. Let $T = \{1, ..., v\}$. If T is meritocratic, it is trivially a weak top group of N and thus the core is not empty. Let us see that if T is egalitarian it is also a weak top group of N. Note first that all agents with productivity below the mean are in their best group. Only agents above the mean could improve. But, since the group is egalitarian, the mean is above the median and thus the group that can improve has a cardinality smaller than $v/2$. But the unproductive group $I = \{v+1, .., n\}$ also has a cardinality smaller that $v/2$. Thus, there is no way of forming a group that can improve upon T. ■

Note, however, that existence of core stable organizational structures is not guaranteed when $3v/2 < n < 2v$, as we have shown in Example 2, where neither the meritocratic group with the greatest mean (the group of the medium productivity agents), nor any of the egalitarian groups with the greatest mean (the high productivity plus two medium productivity agents) are weak top groups of N.

Finally, let us make clear that when $n \geq 2v$, the weak top property is not a necessary condition for the existence of core stable organizational structures. This can be checked in Example 1.

4. Three type societies

In this section we begin to study the benchmark case where agents can be classified into three classes. As we already remarked in the introduction, the study of such cases is standard and productive in many contexts. What we add here is that agents within each class have exactly the same productivity level, which we identify with their type. In Section 5 we extend our results to the case where the three classes can still be clearly identified and yet productivities can differ across individuals within each class.

Formally, in a three type society, (N, λ, v), a generic type is denoted by j, $j \in \{h, m, l\}$, and productivities are $\lambda_h > \lambda_m > \lambda_l$. We denote by H, M, and L be the sets of all high, medium and low type agents respectively, and by n_H, n_M and n_L the cardinality of these sets. The order of individuals of the same type is arbitrary and will have no effect on our results. Note that because of this arbitrariness, any two organizational structures which only differ in the numbering of individuals of the same type are structurally equivalent in the sense that if one is core stable the other will also be. In what follows when we refer to uniqueness of core stable organizational structures, we mean that they are all structurally equivalent.

In Section 4.2 we present a general, necessary and sufficient condition for the existence of stable organizational structures for three type societies. Before that, in Section 4.1, we analyze the special case where $v = n/2$. This case is interesting for several reasons. One is that, in that case, existence is always guaranteed. Moreover, we can then identify and characterize those societies where non segregation is not only possible but in fact is required for stability.

4.1. The case of three types and $v = n/2$

Our initial purpose in this section is to prove that in this case stable organizational structures will always exist. Remember that in Section 3 we already proved that, should there only be one or two types in society, existence is guaranteed. So, we must just prove it for the non-degenerate case where there is at least one agent of each type. To do so, it is useful to concentrate on the segregated partition $(T, N/T)$ where a group of most productive agents of size v is formed, and the rest of agents gather together in a second group. We shall prove that either this structure is in the core, or else a different organizational structure will be core stable, unique and include a non-segregated group.

To distinguish between these two cases, let us classify societies by introducing the distributional characteristics of productivities that will mark the difference between their stable structures. The definition that follows has technical consequences, but we want to emphasize that it covers situations that will plausibly apply in many applications: it requires that the bulk of population be of a medium type, with a few highly productive agents and also some low productivity agents, and imposes some additional constraints on the ability to form meritocratic coalitions involving the three types.

Definition 5. *A society is maximally mixed meritocratic if* $n_H < v/2$, $n_L \leq v/2$, *and* $(\lambda_h + \lambda_m + n_L\lambda_l)/(n_L + 2) \leq \lambda_m$.

In maximally mixed meritocratic societies we can always construct a meritocratic group of cardinality v that contains agents of all three types, all agents of the low type and the highest number of high types allowing for all the preceding characteristics to hold. We call this a *maximally mixed meritocratic group*, and denote it by $M3$. This group can be constructed as follows. Start with all n_L low types, one medium type and one high type. This starting group may not be productive, but the mean of its $\lambda' s$ is below λ_m. Next add as many high types as possible while keeping the mean of the $\lambda' s$ below λ_m. And finally, if the group is not yet productive, fill the set with medium types until reaching size v. Note that $N \backslash M3$ is either an egalitarian group with high and medium type agents, or a meritocratic group with only medium type agents. Remark that an organizational structure that contains a group with the characteristics of $M3$ is non-segregated.

Proposition 4. *(a) In three type societies where $v = n/2$, stable organizational structures always exist.*
(b) If societies are maximally mixed meritocratic, then the structure $(M3, N/M3)$, where $M3$ is non-segregated, is the only stable one.
(c) If societies are not maximally mixed meritocratic, then the segregated organizational structure $(T, N/T)$ is stable, and there is at most another stable structure.

Proof. In fact, proving (b) and (c) implies (a). We start by statement (b).

(b1) To prove that $(M3, N/M3)$ is a stable organizational structure, first notice that medium type agents in $N\backslash M_3$ can only improve upon if they can join an egalitarian group with highest mean. But such superior group must include high type agents from M_3 that are not willing to join since M_3 is meritocratic. High type agents in $N\backslash M_3$ if any, could be better off joining an egalitarian group with greater mean or a meritocratic group. The first case is ruled out by the same argument used for medium type agents. The second is not possible either since, by construction, there is no other meritocratic group that can be formed without using other medium type agents from $N/M3$.

(b2) The proof that $(M3, N/M3)$ is the only core stable structure proceeds as follows.
- We first show that no structure with only one productive group can be core stable. Such productive group would have to be weak top. Candidates to be weak top groups are $G \in E_+(N)$, or $G \in M_+(N)$.

If $G \in E_+(N)$, G has size v, contradicting that the organizational structure includes only one productive group. In a maximally mixed meritocratic society, $n_H < v/2$, $n_L \leq v/2$, and consequently $n_M > v$, which imply that T is egalitarian, $T \in E_+(N)$, and any other $G \in E_+(N)$ is equivalent to T.

If $G \in M_+(N)$, then G contains only medium type agents. This is because any meritocratic group with high type agents has the mean below the productivity of the medium type agents, and $n_M > v$. But groups composed only of medium type agents are never weak top, because its members always prefer to add high types to their group.
- We now concentrate in organizational structures containing two productive groups $(G_1, G_2) \neq (M3, N\backslash M3)$, and prove that there will always be a group blocking (G_1, G_2).

(i) If G_1 and G_2 are both meritocratic, both groups have three types of agents or one of them three types and the other two types, medium and low. In any case, adding the medium type agents to the group with greater mean forms a meritocratic group with increased mean that blocks

(G_1, G_2).

(ii) If G_1 and G_2 are both egalitarian then none of them is T, because $N \backslash T$ is meritocratic. Thus, T blocks (G_1, G_2).

(iii) If G_1 is meritocratic and G_2 is egalitarian, $G_2 \neq T$. Because otherwise, $G_1 = N \backslash T$ and then $M3$ blocks $(T, N \backslash T)$. The group G_2 cannot have three types of agents, because by replacing low types in G_2 by medium types, the mean increases while keeping egalitarianism. This new group will block (G_1, G_2). Thus, G_2 can only contain two types of agents. Since $n_H < v/2$, $n_L \leq v/2$, G_2 contains only high and medium types. Since G_2 is different from $N \backslash M3$ it must contain more high type agents. But then, given the construction of $M3$, we can replace medium type agents in G_1 by high type agents while keeping meritocracy and increasing the mean, and this new group will block (G_1, G_2).

Thus, $(M3, N \backslash M3)$ is the unique core stable organizational structure.

(c1) The existence statement in part (c) follows from the analysis of different possibilities, that we take in turn. If society is not maximally mixed meritocratic, then either $n_H \geq v/2$, or $n_L > v/2$, or $n_H < v/2$ and $n_L \leq v/2$ but $(\lambda_h + \lambda_m + n_L \lambda_l)/(n_L + 2) > \lambda_m$.

- If $n_H \geq v/2$, $T \in M_+(N)$, and therefore is a weak top group. Thus, $(T, N \backslash T)$ is a core stable organizational structure.

- If $n_H < v/2$, but $n_L > v/2$, T can be meritocratic (with three types) or egalitarian (with high and medium types). In the first case $T \in M_+(N)$, and therefore is a weak top group. Thus, $(T, N \backslash T)$ is a core stable organizational structure. In the second case, (T, B) is such that T is egalitarian and B is either egalitarian or meritocratic with just low type agents. In any of the situations (T, B) is clearly a core stable organizational structure.

- Finally, if $n_H < v/2$ and $n_L \leq v/2$ but $(\lambda_h + \lambda_m + n_L \lambda_l)/(n_L + 2) > \lambda_m$, T only contains high types and medium type agents and it is egalitarian, $N \backslash T$ contains only medium and low type agents and is meritocratic. Condition $(\lambda_h + \lambda_m + n_L \lambda_l)/(n_L + 2) > \lambda_m$ implies that high type agents cannot be part of a meritocratic group, thus T is a weak top group of N and $(T, N \backslash T)$ is a core stable organizational structure.

(c2) The proof that, in addition to the segregated partition, there will be at most another stable organizational structure in that case, is in the appendix. ■

We close the section by recapitulating what we have learned, and highlighting some of the main findings about stable structures that basically extend when we allow $n > 2v$.

One first lesson refers to segregation. For societies that are maximally mixed meritocratic, stability implies non-segregation, as proven in Propositions 4. For societies that are not, we can assert for sure that stability holds for the segregated structure $(T, N\backslash T)$, but this is sometimes compatible with the existence of a second stable structure which may be non-segregated.

A second set of remarks refer to the combinations of reward schemes that are compatible within core stable organizational structures. In societies that are maximally mixed meritocratic, at least one of the groups in a stable structure must be meritocratic, while the second group may adopt meritocracy or egalitarianism. In societies where $(T, N\backslash T)$ is stable, each one of the two sets can adopt any of the two distributional criteria. Moreover, note that in this case the resulting distributional criteria are determined by the number of agents of each type that belong to each of the two sets, and not on the exact values of their productivities.

The (almost) uniqueness results in the present section provides the grounds for the use of comparative statics that we have discussed in the Examples of Section 2.

4.2. Three type societies: the general case.

In this subsection we discuss the characteristics of three-type societies where core stable organizational structures exist and also the form that these structures take under different conditions.

We shall distinguish between two sets of societies, that we call structured and unstructured, and prove that the limits between the two indeed determine whether or not core stability can be attained. We can prove that core stable organizational structures will exist in a society if and only if it is structured.

The reader will appreciate that many of the ideas that arose in the preceding subsection do come back, but with some additional complications that were avoided in the case where only two productive groups could be formed.

Since the definition of a structured society is complex, we start by describing its characteristics from two different perspectives.

First, regarding the type of groups that may be part of core stable structures. We'll prove that such structures must either contain T or some meritocratic group G with high type agents. Although this does not provide a full description of the whole structure, it points at a salient group in it. We'll say that core stable partitions must be structured around T or around some meritocratic G, meaning that one of these sets has to be part of the partition and that the rest of society must be able to accommodate the further requirements imposed by overall stability. As a result, stability requires in all cases that some of the high type agents are part of a group where they get their best possible treatment. They will either be all part of the best egalitarian group, when no stable partition can be structured around any meritocratic group containing high types, or else some of them will manage to structure a stable organization around a meritocratic group, where they get paid their full productivity, even if sometimes at the expense of other high type agents.

Second, we can look at the requirements that separate these two types of societies. In order to be unstructured, a society must have a rather special distribution of types. In particular, it must satisfy at least the following requirements: (i) It must be that the number of high type agents is less than $v/2$. Otherwise, they could form a meritocratic group including all of them, and let the remaining members of society, which will now be of at most two types, to organize in a stable manner. (ii) In addition, unstructured societies must contain a number of middle types that is bounded above and below, so that $v \leq n_H + n_M < 2v$. This is because a very small middle class, when coupled with a small high class, cannot de-stabilize a partition structured around T, while a large enough middle class will leave room for T to structure a stable partition again, this time thanks to the fact that the remaining middle type agents not in T will be able to achieve the highest mean meritocratic group, the one formed by medium type agents alone. In the case $n_H + n_M < 3v/2$, unstructured societies must contain a "sufficient" number of low types to allow high type agents to challenge a partition structured around T with a meritocratic group. Finally, (iii) unstructured societies are not able to satisfy medium type agents. Any partition structured around a meritocratic group G with high type agents can always be challenged by some of the medium type agents.

We will write $H(G)$, $M(G)$, and $L(G)$ to denote respectively the high, medium and low type agents in G. The formal definition of a structured society is as follows.

Definition 6. *A three type society is structured if at least one of the following three conditions holds:*

1. N has a weak top group.

2. Either $n - n_L \geq 2v$, or $n - n_L < 3v/2$ and for all G meritocratic such that all $i \in G \cap T$ are better off in G than in T, $0 \leq n - 2v < \#L(G)$.

3. There exists a meritocratic group G_1 with $G_1 \cap H \neq \emptyset$ and $\#(N \backslash G_1) \geq v$ such that:

(a) $\bar{\lambda}_{G_1} \geq \bar{\lambda}_G$ for all meritocratic group $G \subset (G_1 \cup H(G_2) \cup G_3)$ where $G_2 = T(N \backslash G_1)$ and

$G_3 = N\backslash(G_1 \cup G_2)$, and

(b) Either $\#(H \cup M)\backslash G_1 = v$ or $\#(M \cup H(G_2)) < v$, $M \subset G_1$ and $\bar{\lambda}_{T(M \cup (N \backslash G_1))} < \lambda_m$.

A three type society is unstructured if it is not structured, that is, if none of the above conditions holds.

Note that condition 1 is a limited version of the weak top group condition. Recall that the latter is a sufficient condition for the existence of core stable structures in general hedonic games. Here we only need to require the existence of a weak top group of N, the set of all agents. Also remark that, in view of Proposition 1, this condition is an easy one to check. Given its transparency, we do not elaborate any further regarding it. Condition 2, then, specifies that a society may still be structured, this time around T, in the absence of a weak top group for N, provided the set of middle productivity agents is "small enough" or "large enough", in the sense of point (ii) in our preceding discussion. Notice that these cases essentially extend the ideas we discussed when $v = n/2$, for the case where the segregated partition is stable. Similarly, though with some added complication, condition 3 provides conditions for the existence of a stable organizational structure around a non-segregated coalition, in the spirit of the maximally mixed meritocratic societies discussed in subsection 4.1.

Thus, the definition of a structured society is "nested" in the following sense: Condition 1 is a sufficient condition for existence of stable organizational structures. If condition 1 does not hold, condition 2 is sufficient for the existence of stable organizational structures, and finally, if neither condition 1 nor condition 2 hold, condition 3 is sufficient for the existence of stable organizational structures. Furthermore, if none of the conditions hold the core is empty. The following proposition formally states these results.

Proposition 5. *There exist core stable organizational structures for a three type society if and only if the society is structured.*

Proof. *Part 1: Structured societies have core stable organizational structures.*

For each condition assuring a structured society we describe how to construct a core stable organizational structure.

(i) Suppose condition 1 holds, i.e. there exist weak top groups in N. We first argue that there will always be one weak top group W such that $N\backslash W$ contains only two types. This is because

- if $n_H \geq v$, then H is weak top (in fact top), and therefore $N\backslash H$ contains two types of agents, medium and low.

- if $n_H < v$, and T is meritocratic, T is weak top and $N \backslash T$ contains at most two types of agents, medium and low.

- if $n_H < v$, T is egalitarian and weak top, then $N \backslash T$ contains at most two types of agents, medium and low.

- if $n_H < v$, T is egalitarian but not weak top, then any weak top group W must be meritocratic with highest mean. W must contain some high type agents, because all agents in a meritocratic group without high type agents will gain from adding one high type, whether this enlarged set is egalitarian or meritocratic. In addition, W must contain all medium type agents, because if one of them was left out, adding that agent would increase the group mean while keeping meritocracy. Then $N \backslash W$ contains at most two types of agents, high and low.

Let us now construct a core stable structure. Take a weak top group W such that $N \backslash W$ contains only two types. We have just shown that this is always possible. Let W be one of the groups in the organizational structure. Note that because $N \backslash W$ it is composed of only two types, it has a core stable organizational structure; combining this structure with W we obtain a core stable structure for our initial society.

(ii) Suppose that condition 1 does not hold but condition 2 holds. Since condition 1 does not hold, $T = \{1, .., v\}$ is egalitarian, thus $n_H < v/2$.

If $n - n_L \geq 2v$, high and medium types alone can form two productive groups. Let $G_1 = T$, $G_2 = M \backslash T$, and $G_3 = L$. Clearly (G_1, G_2, G_3) is a core stable organizational structure.

If $n - n_L < 2v$, then $n - n_L < 3v/2$ and $0 \leq n - 2v < \#L(G)$ for every meritocratic group G such that all $i \in G \cap T$ are better off in G than in T. Let $G_1 = T$, $G_2 = \{v + 1, .., 2v\}$ (G_2 is an egalitarian group given that $n - n_L < 3v/2$), and $G_3 = N \backslash (G_1 \cup G_2)$ a group of low types. Again (G_1, G_2, G_3) is a core stable organizational structure. This is because the potential blocking group of this structure is a meritocratic group G that contains low type agents. But since low type agents in G_2 are in an egalitarian group, they cannot be part of the blocking, and since $n - 2v < \#L(G)$, for any of those potential meritocratic groups blocking π, low type agents in G_3 are not enough to form the potential blocking group G.

(iii) Last, suppose that condition 1 and 2 fail but condition 3 holds.

First of all note that, because of the failure of 1 and 2, $n_H < v/2$ and $n_H + n_M < 2v$.

Second, because 3 holds, there exists a meritocratic group G_1 with $G_1 \cap H \neq \emptyset$ and $\#(N \backslash G_1) \geq v$ satisfying (a) and (b). Let $\pi = (G_1, G_2, G_3)$ where $G_2 = T(N \backslash G_1)$ and $G_3 = N \backslash (G_1 \cup G_2)$.

If $\#(H \cup M)\backslash G_1 = v$, G_2 is either an egalitarian group with high and medium types or just a meritocratic group with medium type agents if all high type agents are in G_1, and G_3 is a group of low types. If $\#(H \cup M)\backslash G_1 \neq v$, all the medium type agents are in G_1, G_2 is an egalitarian group with high and low types and G_3 is a group of low type agents if any. In both cases, conditions a and b guarantee that π cannot be blocked.

Part 2: Unstructured societies have no core stable organizational structures.
Assume that neither 1 nor 2 nor 3 hold and that a core stable organization structure π exists. Let $G \in \pi$ such that $G \cap H \neq \emptyset$. We show that G cannot be meritocratic, nor egalitarian, nor unproductive, which is a contradiction.

(i) Assume G is meritocratic.

Since condition 1 does not hold, there are no weak top groups in N. Then $n_H < v/2$, because otherwise T would be a meritocratic group and it would be a weak top group of N. Thus, if G is a meritocratic group it must include three types of agents. Since there is no weak top group, $\#N\backslash G \geq v$, because otherwise, if the remaining agents are in an unproductive group, π can be blocked. Apart from G, no other productive group $G' \in \pi$ with three types can be meritocratic. Otherwise the medium type agents in the group with lower average productivity can switch to that other group. This generates a meritocratic new group with a greater average productivity that blocks π. So, if π contains another productive group G' with three types, that G' must be egalitarian and it must contain all the high type agents in $(H \cup M)\backslash G$. If $\bar{\lambda}_{G'} > \lambda_m$, replacing a low type in G' by one of the medium types in G increases the average and keeps egalitarianism, and this later group blocks π. But if $\bar{\lambda}_{G'} \leq \lambda_m$ we contradict that π is core stable as well - since switching one of the medium types from G' to G increases the average in G and keeps meritocracy. Thus, agents in $N\backslash G$ can only be organized in two-types groups, and the high types in $N\backslash G$ are in an egalitarian group. Note also that medium type agents cannot be in a group with just low type agents, because by joining G they increase the mean while keeping meritocracy, and this new group will block π. Thus, π contains $G_2 = T(N\backslash G)$, which is either egalitarian with high and medium types, or meritocratic with just medium type agents (if all high type agents are in G), or egalitarian with high and low types if G contains all the medium agents. In any case, the remaining agents, $N\backslash(G \cup G_2)$ are low type agents.

Since condition 3 does not hold, either (a) or (b) fails.

If (a) fails, a meritocratic group $G' \subset (G \cup H(G_2) \cup G_3)$ where $G_2 = T(N\backslash G)$ and $G_3 =$

$N\backslash(G \cup G_2)$ exists with $\bar{\lambda}_{G'} > \bar{\lambda}_G$. Since only high type agents in G_2 are potentially part of this meritocratic group, G' blocks π.

If (b) fails, then $\#(H \cup M)\backslash G \neq v$. Since, as we argue above, π cannot place medium type agents in a group with just low type agents, then $\#(H \cup M)\backslash G < v$. Thus, all medium type agents are in G, and π organizes $N\backslash G$ with an egalitarian group with high and low types and a group of low type agents alone. If $\#(M \cup H(G_2)) \geq v$, then the group of cardinality v with high types not in G and medium type agents is egalitarian (or meritocratic if only contains medium type agents) and blocks π. If $\#(M \cup H(G_2)) < v$, the average productivity of $T(M \cup (N\backslash G))$ is greater than λ_m, which implies that $T(M \cup (N\backslash G))$ is an egalitarian group which blocks π.

Because of all the above points, high type agents cannot be in a meritocratic group.

(ii) Assume next that G is egalitarian.

Then, since there are no weak top groups and high type agents cannot be in a meritocratic group, it must be that $G = T$. Since condition 2 does not hold, $n - n_L < 2v$ and either $n - n_L \geq 3v/2$ or there exist a meritocratic group G^* such that all $i \in G^* \cap T$ are better off in G^* than in T and $n - 2v \geq \#L(G^*)$.

In the first case, any organizational structure containing T, where agents in $N\backslash T$ are organized in a stable way, is such that $T(N\backslash T)$ is a meritocratic group with medium and low types, and the remaining agents are just low type agents. Since T is not weak top, a meritocratic group G' exist such that all $i \in G' \cap T$ are better off in G' than in T. This meritocratic group contains high type agents in T and medium and low types in $N\backslash T$. Medium type agents in $N\backslash T$ are in a meritocratic group and low type agents are also in meritocratic groups or alone. Then G' blocks π because (1) high type agents in $G' \cap T$ are better off in G' than in T, and (2) medium and low types in G' are better off than in their respective groups because G' has a greater mean.

In the second case, $T(N\backslash T)$ is egalitarian, and the low agents in $T(N\backslash T)$ cannot be used to block π with a meritocratic group. But, since condition 2 fails, then a meritocratic group can be constructed that blocks π. This is because the remaining low types not in T neither in $T(N\backslash T)$ are enough to construct G^*.

(iii) To conclude, assume G is unproductive.

Given that $h \in G$ is very welcome in any group, T blocks π.

Hence, there are no core stable organizational structures. ■

Remark 1. *(a) Note that when $n < 2v$, conditions 2 and 3 in the definition of a structured society* *never hold because they involve restrictions that only apply when more than one group can form. Hence, if $n < 2v$ a society is structured if and only if N has weak top groups. This remark leads us directly to the necessary and sufficient condition for the existence of core stable organizational structures that we already discussed in Proposition 2.*

(b) Also note that, in a structured society that fails to satisfy conditions 1 and 2, the unique stable organization structures are non-segregated. They are structured around a meritocratic group G that may or may not contain all high type agents. If G leaves some high type agents out, these must be organized in an egalitarian group. If G contains all the high type agents, there must be enough medium type agents out of G to form a productive group by themselves. These conditions are the analogue to the maximally mixed meritocrratic property for general three-type societies.

5. Three way clustered societies

In this section we extend our analysis of societies that can be divided into three classes to a much more general case than the one we just considered. We now allow for agents within a class (or cluster) to have different productivities, provided the agents in each class are sufficiently similar, relative to that of agents in other classes, in terms that are made precise in the definition that follows. With some adjustments, we provide a new definition of structured societies within this larger context, and prove that being structured in the extended sense is very much related to the existence of stable organizational structures, which again can be of different forms depending on distributional characteristics. This extension proves that our previous results, based on a simplified model are robust, even if we do not get a full characterization result as we did before.

Definition 7. *A society $\mathcal{S} = (N, \lambda, v)$ is three clustered if there exists a partition of N into three groups $\{H, M, L\}$ (clusters)*[7] *with the following properties:*

C1. For all $h \in H$, $m \in M$, and $l \in L$, $\lambda_h > \lambda_m > \lambda_l$.

C2. For any $J \in \{H, M, L\}$, all segregated productive subgroups of J are meritocratic.

C3. For any J, $J' \in \{H, M, L\}$, $J \neq J'$ such that $\lambda_i < \lambda_j$ for all $i \in J$, $j \in J'$, and for any $S_J \subseteq J$ and $S_{J'} \subseteq J'$ $\lambda_i < \bar{\lambda}_{S_J \cup S_{J'}} < \lambda_j$ for all $i \in S_J$, for all $j \in S_{J'}$.

C4. For all $S_H \subseteq H$, $S_L \subseteq L$ and $j \in M$ and $S_M \subseteq M$, if $\bar{\lambda}_{S_H \cup \{j\} \cup S_L} < \lambda_j$ (resp $> \lambda_j$), then $\bar{\lambda}_{S_H \cup S_M \cup S_L} < \lambda_i$ (resp $> \lambda_i$) for all $i \in S_M$.

[7] When this does not lead to confusion and in order to avoid repetitions we may sometimes refer to those agents belonging to the same cluster as being of the same type. Notice however that unlike in the preceding section this loose way to speak does to imply that two members of a cluster are identical.

Condition $C1$ just requires that clusters must be formed by agents whose productivities are correlative in the natural order, and thus allows to properly speak about the high, the medium and the low cluster. All the agents with the same productivity must belong to the same cluster. Condition $C2$ is an intracluster condition. It always holds if for example productivities of the agents in a cluster are uniformly distributed or have a concave distribution, that is, for any three consecutive agents $i, j, k \in J$ with $\lambda_i \leq \lambda_j \leq \lambda_k$, $\lambda_k - \lambda_j \leq \lambda_j - \lambda_i$. Conditions $C3$ and $C4$ are intercluster conditions. Condition $C3$ requires that there should be enough "distance" between any two clusters. Condition $C4$ requires that the average of productivities for any set containing elements of the three clusters should be "strictly between" clusters. That is, either it belongs to the interval $(\min_{j \in S_H} \lambda_j, \max_{j \in S_M} \lambda_j)$ or to the interval $(\min_{j \in S_M} \lambda_j, \max_{j \in S_L} \lambda_j)$.

The following notation will be useful in what follows. Given a society (N, λ, v) and any set $G \subseteq N$ of cardinality n_G, k_G denotes the maximal number of productive groups of size v in G and $r_G = n_G - k_G v$. Subsets of G are denoted S_G. The partition of the first $k_G v$ elements of G into k_G segregated minimal size productive groups is denoted by $(S_G^1 ... S_G^{k_G})$: that is, $S_G^1 = T(G)$ and $S_G^k = T(G \backslash \cup_{k=1}^{k-1} S_G^q)$.

Remark 2. *Our definition allows for three clustered societies which are degenerate in the sense that some of the clusters may be empty. In these cases, it is easy to prove that core stable organizational structures exist. When only one cluster in non empty, only the intracluster condition $C2$ is operative. And then, the segregated partition of the $k_N v$ most productive agents into k_N meritocratic groups of size v, along with an unproductive group formed by the r_N less productive agents is trivially core stable.*
In societies with two non-empty clusters, say H and L, let R_H be the set that contains the last r_H agents in cluster H and at most the $v - r_H$ most productive agents in cluster L. If $n_L < v - r_H$, R_H is an unproductive group and the structure $(\{S_H^k\}_{k=1}^{k_H}, R_H)$ is core stable. If $n_L \geq v - r_H$, let $\hat{L}$ be the remaining agents in the low cluster, that is, $\hat{L} = L \backslash R_H$. Then $(\{S_H^q\}_{q=1}^{k_H}, R_H, \{S_{\hat{L}}^q\}_{q=1}^{k_{\hat{L}}}, U)$, where U is an unproductive group formed by the $r_{\hat{L}}$ less productive agents is a core stable structure.

We now turn to the non degenerate case with three non empty clusters. Our first result refers to the distribution of agents from the high cluster within any core stable organization.

Proposition 6. *If a three cluster society $\mathcal{S}$ has a core stable structure, then at most $v - 1$ agents in H belong to groups containing agents from other clusters.*

Proof. Let $\hat{\pi}$ be a core stable organizational structure for society $\mathcal{S}$. Denote by S_H the subgroup of the high cluster whose agents are assigned in $\hat{\pi}$ to groups containing individuals from other clusters. Refer to groups containing agents from at least two clusters as mixed groups. Assume that $\#S_H \geq v$. If all mixed groups containing agents from S_H are egalitarian, by condition $C3$ the high type members receive a payoff below their productivity. In this case, the group S_H, which has a greater average, will block $\hat{\pi}$ independently of its regime. If some of the mixed groups containing agents from S_H are meritocratic, we distinguish two cases:

(i) Suppose that there is at least a productive subgroup of S_H which is meritocratic. Then, this subgroup constitutes a blocking group of $\hat{\pi}$, because it is meritocratic and has a greater mean that any of the other groups in $\hat{\pi}$ containing agents from S_H.

(ii) Suppose all productive subgroups of S_H are egalitarian. Consider the meritocratic group in $\hat{\pi}$ containing agents from S_H with the greatest mean. Call this group G. Let $j \in G$ be the agent in G not in S_H with the greatest productivity in G. Form the group $G' = S_H \cup \{j\}$. The group G' is meritocratic because agents in S_H form a majority and, by $C3$, the average of the group is between the productivity of the less productive agent in S_H and λ_j. If $G' \neq G$, then G' is a blocking group of $\hat{\pi}$. If $G' = G$, suppose first that some agents of the high cluster not in S_H are organized in an egalitarian group. This implies that some of those agents are receiving less than their productivity. Add those agents to G'. The new group is meritocratic with a greater mean than G', and will block $\hat{\pi}$. Suppose now that all agents outside S_H are organized in meritocratic groups. Since S_H form an egalitarian group, it is non-segregated nor are some of the groups with high types outside S_H. Order the groups in $H \backslash S_H$ so that the first one is the one that contains the highest productivity agent, the second the one which contain the highest productivity agent among the remaining agents, and so on. Consider the first group in this order which is non-segregated and let i be the agent with the greatest productivity in that group. Form the segregated productive group of cardinality v that contains i as the highest productivity agent. Note that to form this group we could use agents in S_H. Clearly this new meritocratic group will block $\hat{\pi}$.

All the above arguments imply that at most $v - 1$ agents in H belong to groups containing agents from other clusters. ■

In view of Proposition 6 it is important to understand the characteristics of core stable organizational structures in societies with at most $v - 1$ agents in the high cluster.

We first define a condition that is necessary and sufficient for the existence of core stable organizational structures for such societies. It is a natural extension of our precious notion of structured societies.

Definition 8. *A non degenerate three clustered society with $n_H < v$ is structured if the following holds:*

1. N has a weak top group.

2. For all G meritocratic such that all $i \in G \cap T$ are better off in G than in T, either the society $(N \backslash T, \lambda_{N \backslash T}, v)$ has a core stable structure, π_1, such that $\#\{i \in M \backslash T \mid (\text{payoff of } i \text{ in } \pi_1) < \lambda_i\} < \#M(G)$, or $\#\{i \in L \backslash T \mid (\text{payoff of } i \text{ in } \pi_1) \leq \lambda_i\} < \#L(G)$.

3. There exists a meritocratic group G_1 with $G_1 \cap H \neq \emptyset$ and $\#(N \backslash G_1) \geq v$ such that:

(a) $\bar{\lambda}_{G_1} \geq \bar{\lambda}_G$ for all meritocratic groups $G \subset (G_1 \cup H(G_2) \cup G_3)$ where $G_2 = T(N \backslash G_1)$ and $G_3 = L \backslash (G_1 \cup G_2)$.

(b) Either the society $((H \cup M) \backslash G_1, \lambda_{H \cup M) \backslash G_1}, v)$ has a core stable structure with segregated groups all of them productive, or $\#(M \cup H(G_2)) < v$, $M \subset G_1$ and $\bar{\lambda}_{T(M \cup (N \backslash G_1))} < \lambda_m$.

Proposition 7. *A non degenerate three clustered society with $n_H < v$ has a core stable organizational structure if and only if it is structured.*

The proof is similar to that of Proposition 5 and is presented in the Appendix.

Finally, we provide two results regarding core stability in societies with v or more agents in the high cluster. One is a necessary condition and the other a sufficient condition for existence. Both are based on our previous results.

For this purpose, we introduce some additional notation.

Given a non degenerated three cluster society $\mathcal{S}$, let $\mathbf{C}^H$ be the set of core stable structures for (H, λ_H, v), the subsociety formed by the high cluster agents. For any $\pi \in \mathbf{C}^H$, let U^π be the set of unproductive agents in π and let $\mathcal{S}^\pi = (U^\pi \cup M \cup L, \lambda_{U^\pi \cup M \cup L}, v)$. That is: we take those high type agents, U^π, that would be in an unproductive group within a stable organization π of the high cluster, and consider the subsociety, $\mathcal{S}^\pi$, that they would form along with agents in the medium and low clusters.

Proposition 8. *If a three cluster society $\mathcal{S}$ has a core stable structure, then there exists $\pi \in \mathbf{C}^H$ such that subsociety $\mathcal{S}^\pi$ is structured.*

Proof. The proof is a direct consequence of Propositions 6 and 7 and the fact that subpartitions of a stable organization must be stable within their subsociety. ■

Recall that by Remark 2 the subsociety (H, λ_H, v) has at least one core stable structure, namely the segregated partition. Denote it by $\pi^s = (\{S_H^k\}_{k=1}^{k_H}, R_H)$. With this notation, the sufficient condition reads as follows.

Proposition 9. *Consider a three clustered society $\mathcal{S}$. If the subsociety $\mathcal{S}^{\pi^s} = (R_H \cup M \cup L, \lambda_{R_H \cup M \cup L}, v)$ is structured then $\mathcal{S}$ has a core stable organization.*

Proof. Take $\pi^s \in \mathbf{C}^H$, that is $\pi^s = (\{S_H^k\}_{k=1}^{k_H}, R_H)$, and let $\pi(\mathcal{S}^{\pi^S})$ be a core stable structure of $\mathcal{S}^{\pi^c}$. Let us see that $\pi(\mathcal{S}) = (\{S_H^k\}_{k=1}^{k_H}, \pi(\mathcal{S}^{\pi^c}))$ is a core stable structure of $\mathcal{S}$. If a set G blocks $\pi(\mathcal{S})$ it must contain agents from $H \backslash R_H$ and agents from $M \cup L$. But, given conditions $C3$ and $C4$, the high type agents in a mixed group are always worse off than in a meritocratic group with just high type agents (as they are in $\{S_H^k\}_{k=1}^{k_H}$). This holds because the average of productivities in a mixed group is always smaller than the productivity of the less productive agent in the high type cluster. Thus, $\pi(\mathcal{S})$ is core stable. ■

For this general case we do not reach a full characterization result. There is some gap between the necessary and sufficient condition for existence. The necessary condition is not sufficient because the productive groups in the core stable partition of the high cluster may not match consistently with the core stable partition of the rest of society to form an overall stable organization. The sufficient condition is not necessary because the segregated partition of the high cluster need not be the only form to organize those agents within a core stable organization of the whole society.

6. Continuous Populations.

Many papers that relate to ours are formulated in terms of a continuum of agents. This is a natural way to discuss issues that arise in large societies, like those involving taxation issues, jurisdiction formation or the provision of public goods. It is also a convenient way to be able to circumvent technical problems, or to rely on the full force of analytical tools that cannot be used in discrete models. The main body of our paper is written in a form that makes no use of this modeling possibility, but in the present section we provide an analysis of the same issues in terms of a continuous model and for a special case that parallels that of our basic model in the case of Section 4.1., and reaches the same type of qualitative results. Hence, in addition to its own interest, we present that analysis as proof that our essential conclusions are robust to different modeling choices.

Consider a continuous population $N = [0, 1]$ where individuals' productivities are represented by an increasing function $\lambda : [0, 1] \to [0, 1]$. For simplicity we assume that λ is differentiable and take the minimal measure for a productive group to be $v = 1/k$, where $k \geq 2$ is an integer, the maximal number of potential groups.

A concave λ represents a population with an increasing density: more productive individuals occur more frequently. Populations where frequency decreases in productivity are represented by a convex λ.

The average productivity of individuals in a group $A \subset [0, 1]$ of measure $\mu(A)$ is

$$\bar{\lambda}_A = \frac{1}{\mu(A)} \int_{x \in A} \lambda(x) dx.$$

A segregated group of individuals, i.e. an interval $[a, b] \subset [0, 1]$ has median $\lambda_m = \lambda(\frac{b-a}{2})$.

For λ concave $\lambda_m \geq \bar{\lambda}_{[a,b]}$ and therefore all segregated groups are meritocratic. For λ strictly convex $\lambda_m < \bar{\lambda}_{[a,b]}$ and therefore all segregated groups are egalitarian. Hence, it is immediate that under a concave λ - when the distribution of productivities has an increasing density - a partition of the population into segregated groups of minimal size, all of them meritocratic, is core stable.

Yet, because talent is usually a scarce resource, scenarios with a concave λ are uncommon. To represent populations where frequency decreases in productivity, the appropriate representations is a convex λ. Often, however, populations are single peaked around some intermediate productivity, in this case λ is initially concave and eventually becomes convex. We pay special attention to that case in what follows.

We now become more specific and concentrate for the rest of the section on the case where $v = 1/2$.

We let $T = [1/2, 1]$ and $N \backslash T = [0, 1/2)$.

A first immediate result establishes that for any productivity function such that both T and $N \backslash T$ are egalitarian, the segregated partition of minimal size is core stable.

Proposition 10. *If $v = 1/2$ and λ is such that both $T = [1/2, 1]$ and $N \backslash T = [0, 1/2)$ are egalitarian, then $\pi^s = (T, N \backslash T)$ is core stable.*

Proof. Since both groups in the segregated partition are egalitarian, π^s can only be blocked by a meritocratic group containing the individuals that are rewarded below their productivity. But, in each group, since the mean productivity is above the median productivity, those individuals have measure strictly less that $1/4$. Therefore they can not form a productive group. ■

Strict convexity of λ is a sufficient condition for both $[0, 1/2)$ and $[1/2, 1]$ to be egalitarian. However, if $N\backslash T = [0, 1/2)$ is meritocratic and $T = [1/2, 1]$ is egalitarian, π^s is not core stable.

Proposition 11. *If $v = 1/2$ and λ is such that $T = [1/2, 1]$ is egalitarian and $N\backslash T = [0, 1/2)$ is meritocratic, $\pi^s = (T, N\backslash T)$ is not core stable.*

Proof. Consider the group $G = [0, 1/2 - \varepsilon] \cup [1 - \varepsilon, 1]$. Take ε sufficiently small such that $\lambda(1 - \varepsilon) > \bar{\lambda}_T$, which implies that all agents above $1 - \varepsilon$ will be better off in a meritocratic group than in T. The average productivity of G is given by:

$$\bar{\lambda}_G = 2\int_0^{1/2} \lambda(x)dx - 2\int_{1/2-\varepsilon}^{1/2} \lambda(x)dx + 2\int_{1-\varepsilon}^{1} \lambda(x)dx.$$

Since $\lambda(x) \leq 1$, $\int_{1-\varepsilon}^{1} \lambda(x) < \varepsilon$, thus, $\bar{\lambda}_G < \bar{\lambda}_{N\backslash T} + 2\varepsilon$. Given that $N\backslash T$ is meritocratic and $\bar{\lambda}_{N\backslash T} < \lambda(1/4)$, for ε sufficiently small $\bar{\lambda}_G < \bar{\lambda}_{N\backslash T} + 2\varepsilon \leq \lambda(1/4)$. Therefore G is a meritocratic group with and average productivity greater than $\bar{\lambda}_{N\backslash T}$ that blocks π^s. ■

Let us now examine whether other non segregated organizational structures can be core stable. In this analysis, the maximally polarized group P composed by the top and bottom quartiles

$$P \equiv [0, 1/4) \cup [3/4, 1]$$

plays a fundamental role. This group is meritocratic if and only if

$$\lambda(\frac{3}{4}) > \bar{\lambda}_P = 2\left(\int_0^{1/4} \lambda(x)dx + \int_{3/4}^{1} \lambda(x)dx\right).$$

If T is egalitarian and P is meritocratic, a meritocratic group with maximal mean, that is, with mean equal to $\lambda(\frac{3}{4})$, can be constructed.

Lemma 1. *For λ such that T is egalitarian and P is meritocratic there exist a unique $a \in [0, 1/4]$ such that*

$$2\left(\int_0^{1/4-a} \lambda(x)dx + \int_{3/4-a}^{1} \lambda(x)dx\right) = \lambda(\frac{3}{4}).$$

Proof. Let

$$G(z) \equiv 2\left(\int_0^{1/4-z} \lambda(x)dx + \int_{3/4-z}^{1} \lambda(x)dx\right).$$

Since T is egalitarian, $G(1/4) > \lambda(3/4)$; and since P is meritocratic $G(0) \leq \lambda(3/4)$. The function G is increasing in z because $G'(z) = \lambda(3/4 - z) - \lambda(1/4 - z)$ and λ is increasing. Thus, by the intermediate value theorem, there exist a unique $a \in [0, 1/4]$ such that $G(a) = \lambda(\frac{3}{4})$. ■

Now we establish that, when T is egalitarian and P is meritocratic, the non-segregated group $P^a = [0, 1/4 - a) \cup [3/4 - a, 1]$ and its complement constitute a core stable organization.

Proposition 12. *Assume λ is such that T is egalitarian and P is meritocratic. Let $P^a = [0, 1/4 - a) \cup [3/4 - a, 1]$. The organizational structure $\pi^a = (P^a, N \backslash P^a)$ is core stable.*

Proof. Note first that since P^a is constructed so that the average productivity of the group coincides with the median productivity, and the group contains the first quarter of the most productive agents, P^a is a meritocratic group with the highest mean. Note that $N \backslash P^a$ is a segregated group. Assume first that $N \backslash P^a$ is an egalitarian group. Members of $N \backslash P^a$ with a productivity above the mean would prefer to be in a meritocratic group; but they can not construct a blocking meritocratic group because P^a is a meritocratic group with the highest mean. Members of $N \backslash P^a$ with a productivity below the mean would prefer to be in an egalitarian group with a highest mean but they can not use agents in P^a with higher productivity than them because some of those agents would be worse off. Finally, if $N \backslash P^a$ is a meritocratic group then it is impossible to block since a blocking group would have to be meritocratic and include members of P^a. ■

Now, to be specific, we present two examples of situations where the assumptions of Proposition 12 do hold as a result of plausible and interesting specifications of our model.

Remember our discussion regarding the role of concavity or the convexity of the λ distribution on the nature of stable organizational structures. The family of beta distributions provides us with a tractable parametrization that facilitates our discussion of that issue. For example, the beta parametrization $\lambda(x) = x^r$, $r > 0$, represents a family of distributions of productivities with c.d.f. $F(\lambda) = \lambda^{1/r}$, and a positive and monotone density $\frac{1}{r}\ \lambda^{(1/r)-1}$ on $[0, 1]$, which is increasing or decreasing depending on whether $r \leq 1$ or $r \geq 1$. Any $r \in [0, 1]$ generates a concave productivity function and any $r \geq 1$ a convex productivity function.

Let us say that a productivity function beta $\lambda(x) = x^r$ is *moderately convex* if $1 \leq r \leq 4$.

Proposition 13. *If λ is beta moderately convex, then T is egalitarian and P is meritocratic.*

Proof. First note that since $\lambda(x)$ is convex T is obviously egalitarian.

Let $\lambda(x) = x^r$ with $r \geq 1$. It is easy to check that there is $\bar{r} \geq 1$ such that for all $r \in [1, \bar{r}]$ P is meritocratic. Let

$$H(r) = 2\left(\int_0^{1/4} x^r dx + \int_{3/4}^1 x^r dx\right) - (\frac{3}{4})^r.$$

Note first that if $r = 1$, P is meritocratic because the average productivity of P is $1/2$ and the median productivity is $3/4$. Thus $H(1) < 0$. For $r = 10$, P is egalitarian, because the average productivity of P is 0.174 and the median productivity is $\lambda(3/4) \simeq 0.056$. By the intermediate value theorem there is $\bar{r} \in [1, 10]$ such that $H(\bar{r}) = 0$. Since H is increasing in r for all $r \in [1, \bar{r}]$ $H(r) \leq 0$. Therefore, P is meritocratic for all $r \in [1, \bar{r}]$, and $\bar{r}$ can be numerically evaluated at $\bar{r} \simeq 4.1$. ■

For another example, remember that we already observed that it may be natural to assume that populations are single peaked around some intermediate productivity, with λ initially concave and then convex. In that spirit, consider the following class of productivity functions.

We say that λ is *mixed concave-convex symmetric* if and only if (i) $\lambda(x) = 1 - \lambda(1 - x)$, and (ii) $\lambda''(x) < 0$ for all $x \in [0, 1/2)$. Note that these conditions assure that $\lambda(1/2) = 1/2$ and that λ is strictly concave in $[0, 1/2]$ and strictly convex in $[1/2, 1]$. This family describes populations drawn from a probability distribution with a single peaked density, symmetric around $1/2$.

Again, for this special class of distributions, we can state the following result, showing yet another instance where Proposition 12 does apply.

Proposition 14. *If λ is mixed concave-convex symmetric, then T is egalitarian and P is meritocratic.*

Proof. First note that, since $\lambda(x)$ is convex in $[1/2, 1]$, T is obviously egalitarian.

For a mixed concave-convex λ, the mean productivity of P is $1/2$ because

$$\begin{aligned} 2\left(\int_0^{1/4} \lambda(x)dx + \int_{3/4}^1 \lambda(x)dx\right) &= 2(\int_0^{1/4}(1 - \lambda(1 - x))dx + \int_{3/4}^1 \lambda(x)dx) = \\ &= \frac{1}{2} - 2\int_{3/4}^1 \lambda(y)dy + 2\int_{3/4}^1 \lambda(x)dx) = \frac{1}{2} = \lambda(\frac{1}{2}). \end{aligned}$$

Since $\lambda(1/2) < \lambda(3/4)$, P is meritocratic. ■

Summarizing, for a convex productivity function the segregated partition is always core stable. If furthermore λ is beta moderately convex, the non segregated partition containing the polarized group P^a is also core stable. If λ is mixed concave-convex symmetric, the segregated partition is never stable because $T = [1/2, 1]$ is egalitarian and $N \backslash T = [0, 1/2)$ is meritocratic, all core stable structures are non segregated and in particular contain $\pi^a = (P^a, N \backslash P^a)$.

7. Endogenous Effort

Our model has assumed that agents contributions to production are independent of the reward system. This is consistent with our basic purpose in this paper, which is to analyze the consequences of voting for one of two distributional criteria when neither undermines productive efficiency. But we believe that, in fact, reward systems will affect effort whenever effort is costly and agents are allowed to choose how much to contribute to the groups they join. In this section we present a simple model where individual effort decisions are strategic, and agents are still allowed to vote between meritocracy and egalitarianism. Clearly, in such a model, the decision to join a meritocratic group will become favored by the fact that, under this reward scheme, the most productive workers will be willing to exert more effort. We can show that, even within this more elaborate version of the model, our basic conclusion that different regimes can coexist at equilibrium still holds. Hence, we can interpret our basic model as one that gives the most advantage to the emergence of egalitarianism, but whose main results persist after the productive benefits of meritocracy are taken into account.

Here is the model. Given a society (N, λ, v) we assume that a group G with cardinality $g \geq v$, produces $\sum_{i \in G} \lambda_i e_i$, where e_i is the voluntary effort of agent i, which has (individually incurred) cost $\frac{1}{2} e_i^2$. That is, if agent i exerts effort e_i she obtains:

$$\pi_i^M = \lambda_i e_i - \frac{1}{2} e_i^2, \text{ if } G \text{ is meritocratic, and}$$

$$\pi_i^E = \frac{\sum_{j \in G} \lambda_j e_j}{g} - \frac{1}{2} e_i^2, \text{ if } G \text{ is egalitarian.}$$

We assume that for each group G and each reward regime agents choose efforts simultaneously and non-cooperatively. Hence endogenous efforts will be determined by the unique Nash equilibrium of this non cooperative game. It is easy to check that the Nash equilibrium choice of efforts are as follows.

In an egalitarian group individuals have strong incentives to free ride; they exert effort only in a fraction $1/g$ of their productivity $e_i^E = (\lambda_i / g)$. Hence, the payoffs from membership in egalitarian group G are

$$\pi_i^E = \frac{\sum_{j \in G} \lambda_j^2}{g^2} - \frac{\lambda_i^2}{2g^2}.$$

On the other hand, in a meritocratic group individuals exert effort equal to their productivity $e_i^M = \lambda_i$. Hence, the payoffs from membership in any meritocratic group are

$$\pi_i^M = \frac{\lambda_i^2}{2}.$$

Preferences regarding meritocracy and egalitarianism inside each productive group are a bit more complex than in the baseline model. Agent $i \in G$ prefers meritocracy rather than egalitarianism if and only if $\pi_i^M \geq \pi_i^E$, or equivalently,

$$\frac{\lambda_i^2}{2} \geq \frac{\sum_{j \in G} \lambda_j^2}{g^2 + 1}. \tag{7.1}$$

Additionally, only if necessary to compare two groups with the same regime and identical payoff; the lexicographic preference for greater per capita production applies.

Consider now the internal vote inside each group. Note that if the median member of group G prefers meritocracy to egalitarianism, then all agents with a greater productivity share this preference. Hence, the median productivity member of each group remains decisive voter of the group in the present set up.

Meritocracy prevails more often than in the baseline model, since condition (7.1) may hold for a median with productivity $\lambda_{m(G)} \leq \overline{\lambda}_G$. However the qualitative results of our baseline model are robust. In particular, scenarios where stable organizations structures deliver non-segregated groups and heterogeneous distribution regimes still exist.

For three-type societies where $n = 2v$ the conditions analogous to the "maximally mixed meritocratic societies" that deliver a non-segregated structure in the core are the following:

1. $n_H < v/2$, $n_L \leq v/2$, and $(\lambda_M/\lambda_H)^2 < (2n_H)/((v-1)^2 + 2n_H)$, i.e., T is egalitarian and $N \backslash T$ is meritocratic, and
2. $\lambda_M^2/2 \geq (\lambda_H^2 + \lambda_M^2 + n_L\lambda_L^2)/((n_L+2)^2+1)$, i.e. T is not weak top.

Example 7 is a society where the stable organization is non-segregated and different groups select different regimes.

Example 7. *Endogenous effort and a society with stable non-segregated organizations and different regimes.*

Let $N = \{1, ..., 14\}$, $\lambda = (13, 13, 13, \sqrt{8}, \sqrt{8}, \sqrt{8}, \sqrt{8}, \sqrt{8}, \sqrt{8}, \sqrt{8}, \sqrt{8}, 0, 0, 0)$, $v = 7$.

T *is egalitarian (because* $\lambda_M^2/2 = 4 < 7.1 = (\sum_{j\in T} \lambda_j^2)/(g^2+1)$*);* $N\backslash T$ *is meritocratic (because* $\lambda_M^2/2 = 4 > 3.2 = (\sum_{j\in T} \lambda_j^2)/(g^2+1)$*) with an average production equal to* 4.5714*. But* $(T, N\backslash T)$ *is not stable because the group of agents with productivities* $(13, \sqrt{8}, \sqrt{8}, \sqrt{8}, 0, 0, 0)$ *(where the three medium type agents are from* $N\backslash T$*) is a meritocratic group with an average production of* 27.571 *that blocks* $(T, N\backslash T)$*.*

The structure $\{(1, 4, 5, 6, 12, 13, 14), (2, 3, 7, 8, 9, 10, 11)\}$ *where the first group is meritocratic and the second is egalitarian is (uniquely) stable.*

8. Concluding Remarks

We have presented a very simple model of group formation where people are driven to cooperate by a minimal size requirement, and choose their reward schemes by majority. This model is able to generate a variety of interesting stylized facts that are under examination in different strands of literature, through more complex formulations. Societies, in equilibrium, can generate partitions where meritocracy and egalitarianism co-exist, and where some groups are non segregated while other still gather agents of the same types. We do not claim that the features of our model can be immediately transposed to reality. But they certainly show that one can get a head start in explaining the simultaneous existence of a rich variety of social configurations within societies that choose rewards schemes by vote.

We also want to emphasize that the model is simple to describe, but complex to analyze. We made an effort to provide rather general existence and characterization results in order to clearly establish that the variety of possible stable arrangements that we obtain are not the result of some pathological productivity distributions, but may arise in rich, natural sets of societies.

Before discussing possible extensions, let us comment on the sensitivity of our results to other possible specifications of the model. We have already shown, in Sections 6 and 7, that going to a continuum of agents or introducing the possibility of conditioning effort to rewards, do not alter our conclusions. The reader may also wonder whether the assumption that ties among different groups that provide the same reward are broken in favor of the highest mean productivity group plays any essential role. We claim that our main results would be very similar if those ties were left unbroken. Indeed, our tie breaking assumption makes agents' preferences a bit more demanding and restricts the set of potential core stable organizational structures in some profiles, relative to those that would arise if ties would not be broken. But the frontiers that we establish between societies admitting stability or not remain essentially the same with one exception. Specifically, existence in our three clustered societies without the tie breaking would be guaranteed whenever $n = kv$ for all natural numbers k, while in our case the result is only true when $k = 2$. This is because, in those societies, the segregated partition would always belong to the core.

A second assumption in our model is that agents must chose between only two reward systems. We could have derived the same results by enlarging the set of potential choices to admit any convex combination of these two principles, since in fact agents will always chose one of the two extreme points in that continuum. Our reward systems can be seen as resulting from a model of tax choice where a proportional tax t is levied and its proceeds are equally distributed: egalitarianism corresponds to the case $t = 1$ and meritocracy arises when $t = 0$, since again voters will always favor one of these two extreme cases as their best choice.

Our model admits many other potential extensions. A natural one is to model the externality resulting from cooperation with other agents in other ways. Here the monetary reward is only supplemented lexicographically with some preference to belong to a group with the highest mean, given the same payment. But we could think of stronger impacts to be received from cooperating with others, ones where the prestige of working along with highly productive agents may lead to accept lower pays that the ones one can get in less productive groups. Exploring the combinations between material and subjective rewards would certainly be a next step in understanding the interaction between group formation and the choice of distributional criteria. All of these extensions seem promising, and none of them appears to be trivial.

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9. Appendix

Proof of Proposition 1**.** The proof consists of two parts.

Part 1: Weak top groups of G are congruent groups of G and must belong to either $M_+(G)$ or to $E_+(G)$.

If W is a weak top group of G then it is a congruent group of G.

Next we show that if G has a weak top group, W, then $\bar{\lambda}_W \geq \bar{\lambda}_S$ for all $S \subseteq G \backslash W$. Suppose on the contrary that there is a group $S \subseteq G \backslash W$ such that $\bar{\lambda}_W < \bar{\lambda}_S$. Suppose first that there is an agent $i \in W$ such that $\bar{\lambda}_W \leq \lambda_i < \bar{\lambda}_S$. Let $S' = S \cup \{i\}$. Since $\lambda_i < \bar{\lambda}_S$, the mean productivity of group S' will be bigger than the productivity of i, $\lambda_i < \bar{\lambda}_{S'}$. Thus, agent i, independently of the regime will be better off in S' than in W, in contradiction with W being a weak top group. If there is no agent $i \in W$ such that $\bar{\lambda}_W \leq \lambda_i < \bar{\lambda}_S$, we distinguish two cases: in the first one we suppose that W is egalitarian and in the second we suppose that W is meritocratic.

If W is egalitarian, since no agent $i \in W$ exists such that $\bar{\lambda}_W \leq \lambda_i < \bar{\lambda}_S$, then an agent $i \in W$ exists such that $\bar{\lambda}_S \leq \lambda_i$. Let $S' = S \cup \{i\}$, note first that $\bar{\lambda}_W < \bar{\lambda}_{S\cup\{i\}} \leq \lambda_i$. So, independently of the regime of S', agent i will be better off in S' than in W, in contradiction with W being a weak top group.

If W is meritocratic, since no agent $i \in W$ exists such that $\bar{\lambda}_W \leq \lambda_i < \bar{\lambda}_S$, the median productivity of W is above $\bar{\lambda}_S$. Let $\lambda_{med}(W)$ be this median productivity. Let $i \in W$ such that $\lambda_i < \bar{\lambda}_W$. Suppose first that there is an agent $j \in S$ such that $\lambda_i < \lambda_j \leq \lambda_{med}(W)$. Let $W' = (W \backslash \{i\}) \cup \{j\}$. Note that since the productivities of agents i and j are both below the median productivity of W, replacing in W agent i by agent j does not change the median but increases the average. Thus, all agents in $W' \cap W$ are better off in W' than in W, in contradiction with W being a weak top group. Finally, if there is no an agent $j \in S$ such that $\lambda_i < \lambda_j \leq \lambda_{med}(W)$, then there is an agent $j \in S$ such that $\lambda_j < \lambda_i < \bar{\lambda}_W$. Let $S' = (S \backslash \{j\}) \cup \{i\}$, $\bar{\lambda}_{S'} > \bar{\lambda}_S > \bar{\lambda}_W > \lambda_i$. Thus, independently of the regime of S', agent i will be better off in S' than in W, in contradiction with W being a weak top group.

Suppose now that the weak top group is meritocratic but does not belong to $M_+(G)$. Note first that $W \cap M_+ = \emptyset$ for all $M_+ \in M_+(G)$, because otherwise, all agents in $W \cap M_+$ would strictly prefer M_+ to W contradicting that W is a weak top group. Since $W \cap M_+ = \emptyset$, our previous reasoning applies, and therefore $\bar{\lambda}_W \geq \bar{\lambda}_M$. But then $W \in M_+(G)$, a contradiction. The same argument applies if W is an egalitarian group.

Part 2: If a set in $M_+(G)$ or in $E_+(G)$ is a congruent group of G then it is a weak top group of G.

Suppose $M_+ \in M_+(G)$ is a congruent group of G. If M_+ is a segregated group with the best productivity agents in G, it is clearly a weak top group of G. If it is not of the preceding form, suppose that M_+ is not a weak top group of G. Since it is congruent but not weak top, there is no subgroup of agents in M_+ for which M_+ is the best group. This implies that the most productive agent in G is not in M_+, and for the most productive agent in M_+ there is an egalitarian group E which is preferred to M_+. But then all agents in $E \cap M_+$ would be better off in E, in contradiction with M_+ being congruent.

Suppose finally that $E_+ \in E_+(G)$ is a congruent group of G. If E_+ is a segregated group with the best productivity agents in G, it is clearly a weak top group of G. If it is not of the preceding form, suppose that E_+ is not a weak top group of G. Since it is congruent but not weak top, there is no a subgroup of agents in E_+ for which E_+ is the best group. But note that for the less productive agent in this group E_+ is always its best set, a contradiction. ■

Proof of part (c2) of Proposition 4

Case 1. Assume $n_H \geq v/2$.

There may exist a second core stable structure if (i) $G \in E_+(N)$, $\#G > v$, and G is a weak top group, or (ii) $G \in E_+(N)$, $\#G = v$, and $N \backslash G$ is also egalitarian[8].

If (i), since G is a weak top group, $(G, N \backslash G)$ is core stable and only G is productive.

If (ii), since both G and $N \backslash G$ are egalitarian, $(G, N \backslash G)$ is a core stable organizational structure with two productive groups. To see that, note that no group can block $(G, N \backslash G)$ because such group would have to be meritocratic and thus formed by agents that are receiving less than their productivity in $(G, N \backslash G)$. Given that both groups in $(G, N \backslash G)$ are egalitarian, those agents are the ones whose productivity is above the mean of the group, and since the mean is above the median, they are less than $v/2$ in each group. Hence they cannot form a productive group blocking $(G, N \backslash G)$.

Let us see that, apart from this possible second core stable structure, there can be no other.

[8] This last situation can only happen if $v_L > v/2$. To see this, note that, since T is meritocratic, high type agents have to be distributed between G and $N \backslash G$. Furthermore, let us see that all medium type agents have to be in G. If $\bar{\lambda}_G < \lambda_m$, the median agent is a low type agent, and $N \backslash G$ has to contain three types. Adding a high, a medium, and a low type agent to G from $N \backslash G$ will create a new egalitarian group of higher mean, contradicting that $G \in E_+(N)$. If $\bar{\lambda}_G \geq \lambda_m$, adding a high and a medium type to G from $N \backslash G$ will create a new egalitarian group of higher mean. Again, this contradicts that $G \in E_+(N)$. Thus, G contains all the medium type agents. Therefore, for $N \backslash G$ to be egalitarian, $v_L > v/2$.

In structures $(P, N\backslash P)$ where only P is productive, if P is not weak top, there will exist a productive group G such that all $i \in P \cap G$ will be better off in G than in P. Since all $i \in (N\backslash P) \cap G$ are getting zero in $N\backslash P$, they will also be better off in G. Thus, G will block $(P, N\backslash P)$. Hence, P has to be weak top, and the unique candidate in this case is the one described in (i).

Finally, let us show that any structure (G_1, G_2) with two productive groups different from $(T, N\backslash T)$ and the one considered in case (ii) will be unstable.

(1) If G_1 and G_2 are meritocratic, it is blocked by T which is also meritocratic.

(2) If G_1 is meritocratic and G_2 is egalitarian we distinguish two cases.

- If all the high type agents are in G_1, G_2 can only contain medium and low types, and since it is egalitarian $\bar{\lambda}_G < \lambda_m$. But then, adding a medium type agent from G_2 to G_1 creates a new meritocratic group of higher mean than G_1 which blocks (G_1, G_2).

- If the high type agents are split between G_1 and G_2, we can add all missing high type agents to G_1 and drop enough non high types in G_1 to create a new group of size v. This new group will still be meritocratic, have a higher mean than G_1, and block (G_1, G_2).

(3) If G_1 and G_2 are egalitarian, neither G_1 nor G_2 are in $E_+(N)$. Thus, any egalitarian group $G \in E_+(N)$ will block (G_1, G_2).

Case 2. Assume $n_H < v/2$ and $n_L > v/2$.

In this case, T can be either egalitarian or meritocratic.

Case 2a. Suppose first that T is meritocratic.

Since $n_H < v/2$, T has three types of agents and consequently $N\backslash T$ is the meritocratic group with just low types, which implies that $n_L > v$.

As in Case 1, a second core stable structure may exist if (i) $G \in E_+(N)$, $\#G > v$, and G is a weak top group, or (ii) $G \in E_+(N)$, $\#G = v$, and $N\backslash G$ is also egalitarian[9]. The same argument as in Case 1 applies.

Let us see that, apart from this possible second core stable structure, there can be no other.

As explained in Case 1, structures $(P, N\backslash P)$ where only P is productive are core stable if and only if P is weak top. The unique candidate in this case is the one described in (i).

Finally, let us show that any structure (G_1, G_2) with two productive groups different from $(T, N\backslash T)$ and the one considered in case (ii) will be unstable.

[9] Note that since T is meritocratic, this situation can only happen if G contains all the high type agents and $v - v_H$ low type agents and it should be such that adding a medium type changes the regime. This structure only exists if $v_H = v/2 - 1$ and $v_M < v/2$.

The arguments in (a) and (b) in Case 1 apply here.

In the case that G_1 is meritocratic and G_2 is egalitarian, G_2 must contain at least two types of agents. If G_2 contains high type agents, replacing a low type agent in G_1 by a high type agent will create a new meritocratic group G (because T is meritocratic) of higher mean than G_1 which blocks (G_1, G_2). The same kind of argument will apply if G_2 does not contains high type agents but contains medium type agents.

Case 2b. Suppose that T is egalitarian. Since $n_H < v/2$, T has two or three types of agents and consequently $N\backslash T$ is either egalitarian with medium and low types or meritocratic with only low type agents. There may exist a second core stable structure if (i) $G \in M_+(N)$, $\#G > v$, and G is a weak top group, or (ii) if $G \in M_+(N)$, $\#G = v$, $N\backslash G$ is egalitarian, $(N\backslash G) \cap M = \emptyset$, and the mean productivity of the group is below λ_m.

If (i), since G is a weak top group, $(G, N\backslash G)$ is core stable and only G is productive.

If (ii), since the mean productivity of $N\backslash G$ is below λ_m, $(G, N\backslash G)$ is core stable. There is no possibility of blocking because a potential blocking group should contain medium type agents. Since they are in a meritocratic group with the greatest mean, they will only participate in an egalitarian group with mean above their productivity. But this is not possible.

Let us see that, apart from this possible second core stable structure, there can be no other.

As explained in Case 1, structures $(P, N\backslash P)$ where only P is productive are core stable if and only if P is weak top. The unique candidate in this case is the one described in (i).

Finally, let us show that any structure (G_1, G_2) with two productive groups different from $(T, N\backslash T)$ and the one considered in case (ii) will be unstable.

(a) If G_1 and G_2 are both egalitarian, it is blocked by T which is also egalitarian.

(b) If G_1 and G_2 are both meritocratic, and neither G_1 nor G_2 are in $M_+(N)$, any meritocratic group $G \in M_+(N)$ will block (G_1, G_2). If one of them belongs to $M_+(N)$ (let us say $G_1 \in M_+(N)$), since $n_H < v/2$ and $n_L > v/2$, both G_1 and G_2 contains medium type agents. Suppose that $\bar{\lambda}_{G_1} \geq \bar{\lambda}_{G_2}$: then adding a medium type agent from G_2 to G_1 creates a new meritocratic group of higher mean than G_1 which blocks (G_1, G_2).

(c) If G_1 is meritocratic and G_2 is egalitarian, G_1 may contain agents of two or three types. In the first case they must be medium and low types with a majority of medium types. Thus, G_2 contains low type and high type agents and (possibly) medium types. In any case, $T \in E_+(N)$ blocks (G_1, G_2). If G_1 contains three types, G_2 can contain two or three types (with low and medium types for sure in both cases). If $\bar{\lambda}_{G_2} < \lambda_m$, adding a medium type agent from G_2 to G_1 creates a new meritocratic group of higher mean than G_1, which blocks (G_1, G_2). If $\bar{\lambda}_{G_2} > \lambda_m$, replacing a low type in G_2 with a medium type from G_1 creates a new egalitarian group of higher mean than G_2, which blocks (G_1, G_2).

Case 3. Assume that $n_H < v/2$, $n_L \leq v/2$, and $(\lambda_h + \lambda_m + n_L \lambda_l)/(n_L + 2) > \lambda_m$.

Note that the meritocratic group with the greatest mean in this case is M, which is not a weak top group. Thus, no other organizational structure with only one productive group can be core stable.

Let us show that any structure (G_1, G_2) with two productive groups different from $(T, N \backslash T)$ will be unstable.

(a) Note that G_1 and G_2 cannot be both meritocratic, since there is no meritocratic group that contains high type agents.

(b) If G_1 and G_2 are both egalitarian, it is blocked by T which is also egalitarian.

(c) If G_1 is meritocratic and G_2 is egalitarian, G_1 can only contain medium and low types or only medium type agents, but since this group is different from $N \backslash T$, G_2 must contain low type agents also. Note that since G_2 is egalitarian and low types do not constitute a majority, $\bar{\lambda}_{G_2} > \lambda_m$.

Replacing in G_2 low type agents by medium type agents from G_1 will create a new egalitarian group of higher mean than G_2 which blocks (G_1, G_2). ■

Proof of Proposition 7. *Part 1: Structured societies* with $n_H < v$ *have core stable organizational structures.*

For each condition assuring a structured society we describe how to construct a core stable organizational structure.

(i) Suppose condition 1 holds, i.e. there exist weak top groups in N. Let W be one of those weak top groups. Note first that $N \backslash W$ only contains agents from at most two clusters. This is because either $W = T$ and then $N \backslash T \subset M \cup L$, or W is a meritocratic group with agents from the three clusters. In the latter case, since W is a meritocratic group with maximal average productivity it is necessary that $M \subset W$ and then $N \backslash W \subset H \cup L$. Hence, by Remark 2, the two-type society $N \backslash W$ has a core stable organizational structure. The groups in that structure plus W constitute a core stable organizational structure for N.

(ii) Suppose that condition 1 does not hold but condition 2 does. Since condition 1 does not hold, T is egalitarian, and $(N \backslash T, \lambda_{N \backslash T}, v)$ is a two cluster society. Hence by Remark 2, $(N \backslash T, \lambda_{N \backslash T}, v)$

has a core stable organizational structure π_1. Then $\pi = \{T, \pi_1\}$ is a core stable organization of N because any group G potentially blocking π must be meritocratic and include agents from every cluster, and either some $i \in M \cap G$ is worse off in G than in π_1 (if $\#\{i \in M \backslash T \mid$ (payoff of i in $\pi_1) < \lambda_i\} < \#M(G))$, or else some $i \in L \cap G$ is worse off in G than in π_1 (if $\#\{i \in L \backslash T \mid$ (payoff of i in $\pi_1) < \lambda_i\} < \#L(G))$.

(iii) Last, suppose that conditions 1 and 2 fail but condition 3 holds.

There exists a meritocratic group G_1 with $G_1 \cap H \neq \emptyset$ and $\#(N \backslash G_1) \geq v$ satisfying a and b. Without loss of generality suppose that $i \in G_1 \cap M$ are the agents with the lowest productivity in M (note that if this is not the case, we can always replace each of the medium type agents in G_1 by one less productive medium type agent without changing the above characteristics of G_1). Also without loss of generality, suppose that all $i \in G_1 \cap L$ are consecutive with the greater productivities in L compatible with G_1 being meritocratic. Suppose first that society $((H \cup M) \backslash G_1, \lambda_{(H \cup M) \backslash G_1}, v)$ has a core stable structure with segregated groups, all of them productive. Let $\pi((H \cup M) \backslash G_1)$ be this structure. Consider the following organizational structure of N : the first group is G_1, then all the groups in $\pi((H \cup M) \backslash G_1)$ and finally the core stable structure of the remaining low type agents, $\pi(L \backslash G_1)$. This structure is stable given conditions (a) and (b). Otherwise, if such core structure $\pi((H \cup M) \backslash G_1)$ does not exist, we consider the structure formed by G_1 that contains all the medium type agents (recall that since $\#(M \cup H(G_2)) < v$, medium type agents cannot form a productive group on their own), by $T(N \backslash G_1)$ that contains high and low types, and finally by the core stable structure of the remaining low type agents, $\pi(L \backslash (G_1 \cup T(N \backslash G_1))$. Again, conditions (a) and (b) guarantee that this is a core stable organizational structure for N.

Part 2: Unstructured societies with $n_H < v$ have no core stable organizational structures.

Assume that neither 1 nor 2 nor 3 hold and that a core stable organization structure π exists. Let $G \in \pi$ such that $G \cap H \neq \emptyset$. We show that G cannot be meritocratic, nor egalitarian, nor unproductive, which is a contradiction.

Assume G is meritocratic, let us see that the negation of conditions 1 and 3 lead to a contradiction.

Since condition 1 does not hold, there are no weak top groups in N. Then $n_H < v/2$, because otherwise T would be a meritocratic group and it would be a weak top group of N. Thus, if G is a meritocratic group it must include agents from the three clusters (by $C3$). Since there are no weak top groups, then $\#N \backslash G \geq v$, because otherwise, the remaining agents are in an unproductive group and π can be blocked. Apart from G, no other productive group $G' \in \pi$ with agents from the three clusters can be meritocratic. Otherwise, given $C4$, an $i \in M$ in the group with lower average productivity could switch to the other and increase the average productivity while keeping meritocracy, and this new group would block π. So, if π contains another productive group G' with three types, it must be egalitarian and it must contain all $i \in H\backslash G$. If $\overline{\lambda}_{G'} > \lambda_m$ for some $m \in M$, replacing an agent from L in G' by one from M in G increases the average and keeps egalitarianism, and this later group blocks π (given that $C4$ implies that $\overline{\lambda}_{G'} > \lambda_j$ for all $j \in G' \cap M$). But if $\bar{\lambda}_{G'} \leq \lambda_m$ for some $m \in M$, we contradict that π is core stable as well - since switching one of the agents in M from G' to G increases the average in G and keeps meritocracy. Thus, agents in $N\backslash G$ can only be organized in groups with agents from one or two clusters, and all $i \in H\backslash G$ are in an egalitarian group. Note also that an agent $i \in M \cap (N\backslash G)$ cannot be in a group that does not contain agents from H, because by joining G they increase the mean while keeping meritocracy, and this new group will block π. If $M\backslash G \neq \emptyset$, π contains $G_2 = T(N\backslash G)$ which is egalitarian with agents from H and M, or meritocratic with just agents form M (if $H \subset G$). If there are still more agents in M, they are organized in segregated meritocratic groups with just medium type agents. Note that they cannot be organized in egalitarian groups because the agents in those groups that receive a payoff below their productivity by joining G will increase the mean while keeping meritocracy. The rest of society is composed by agents from L. If $M\backslash G = \emptyset$, π contains $G_2 = T(N\backslash G)$ which is egalitarian with $T(N\backslash G) \subset H \cup L$, and again the remaining society is composed by agents from L. Since condition 3 does not hold, either (a) or (b) fails:

-If (a) fails, a meritocratic group $G' \subset (G \cup H(G_2) \cup G_3)$ where $G_2 = T(N\backslash G)$ and $G_3 = L\backslash(G_1 \cup G_2)$ exists with $\bar{\lambda}_{G'} > \bar{\lambda}_G$. Note that G' blocks π.

-If (b) fails, the society $((H \cup M)\backslash G_1, \lambda_{H\cup M)\backslash G_1}, v)$ cannot be organized in a segregated stable way with all groups productive for any meritocratic group G_1. Since, as we argued above, π cannot place $i \in M$ in groups without agents from H, it must be that $\#(H \cup M)\backslash G < v$. Thus, $M \subset G$, and π organizes $N\backslash G$ with an egalitarian group $E \subset H \cup L$ such that $H\backslash G \subset E$, and the rest of low type agents are organized in an stable way. If $\#(M \cup H(G_2)) \geq v$, then the group of cardinality v containing all $i \in H\backslash G$ and some medium type agents is egalitarian (or meritocratic if it only contains medium type agents) and blocks π. If $\#(M \cup H(G_2)) < v$, the average productivity of $T(M \cup (N\backslash G))$ is greater than λ_m for some $m \in M$, which implies that $T(M \cup (N\backslash G))$ is an egalitarian group which blocks π.

All the above points imply that a meritocratic G containing high type agents cannot be part of a core stable organizational structure of N.

Assume next that G is egalitarian. Let us see that the negation of conditions 1 and 2 leads to a contradiction.

Since there are no weak top groups and high type agents cannot be in a meritocratic group, it must be that $G = T$. Since condition 2 does not hold, any possible stable organization of the society $(N \backslash T, \lambda_{N \backslash T}, v)$ is such that $\#\{i \in M \backslash T \mid (\text{payoff of } i \text{ in } \pi_1) < \lambda_i\} \geq \#M(G)$, or $\#\{i \in L \backslash T \mid (\text{payoff of } i \text{ in } \pi_1) \leq \lambda_i\} \geq \#L(G)$. Thus, the medium and low types necessary to form the meritocratic group that would challenge T are available. This group will block π.

To conclude, assume G is unproductive. But $h \in G$ is very welcome in any meritocratic group (even if that changes the regime), and if there are no meritocratic groups, T blocks π.

Hence, there are no core stable organizational structures. ■