PS 6

GB1 (1) Prove that:

If $v_i(p,w_i) = a_i(p) + b(p)w_i \quad \forall i$

Then the set of consumers exhibit parallel, straight wealth expansion path.

wealth and paths are straight (for each consumer $\frac{\partial x_{\ell i}}{\partial w_i}$ is a constant for each good ℓ) and parallel (all consumers have parallel paths) iff

$$\forall \ell, \forall i,j, \forall w_i, w_j \quad \frac{\partial x_{\ell i}(p, w_i)}{\partial w_i} = \frac{\partial x_{\ell j}(p, w_j)}{\partial w_j}$$

Recall that Roy's identity (3.G.4) allow us to link x and v:

$$\forall \ell, \quad x_\ell(p,w) = - \frac{\partial v(p,w_i)/\partial p_\ell}{\partial v(p,w_i)/\partial w} \quad \text{at } (p,w) \gg 0$$

Thus: $x_{\ell i}(p,w_i) = -\left(\frac{\partial a_i(p)}{\partial p_\ell} + \frac{\partial b(p)}{\partial p_\ell} w_i\right) / b(p)$

It follows: $\frac{\partial x_{\ell i}(p,w_i)}{\partial w_i} = - \frac{1}{b(p)} \cdot \frac{\partial b(p)}{\partial p_\ell}$

which does not depend on i (and in particular on w_i), namely it is constant for all consumers and wealth level.

(2) Show also that $e_i(p,u_i) = c(p)u_i + d_i(p)$

First notice that for $u_i = v_i(p,w_i)$, $e_i(p,u_i) = w_i$, thus.

$$e(p,u_i) = \frac{1}{b(p)}\left(v_i(p,w_i) - a_i(p)\right) = \frac{1}{b(p)} u_i - \frac{a_i(p)}{b(p)}$$

We can take $c(p) = \frac{1}{b(p)}$ and $d_i(p) = -\frac{a_i(p)}{b(p)}$

1.C.3

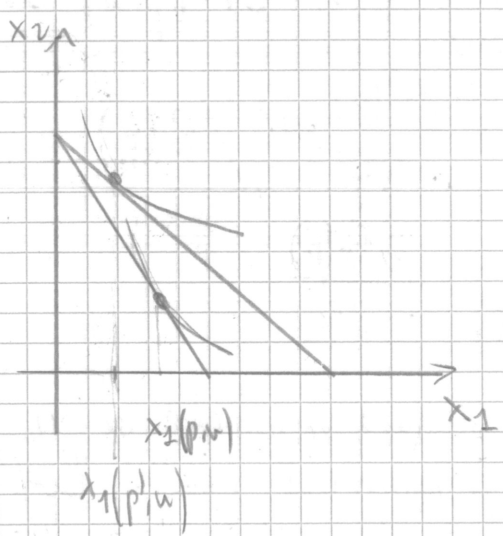
ULD property (uncompensated law of demand):

$$(p'-p)\cdot[x(p',w)-x(p,w)]\leq 0$$

for any p, p', w, with strict inequality if $x(p,w)\neq x(p',w)$

Suppose good 1 is a giffen good at price p. Then surely ULD is not satisfied. In fact for a small increase in price p_1 the demand increases. Let $p'_1 = p_1+\varepsilon$ and $p' = (p'_1, p_2, \dots, p_L)$. Then

$$(p'-p)\cdot[x(p',w)-x(p,w)] = \varepsilon(x_1(p',w)-x_1(p,w)) > 0$$



PS 7

1) a) Inaction: $0 \in Y$

Convexity: $y, y' \in Y \Rightarrow \lambda y + (1-\lambda) y' \in Y \quad \forall \lambda \in [0,1]$

NON increasing return to scale: $y \in Y \Rightarrow \alpha y \in Y \quad \forall \alpha \in [0,1]$

Inaction and convexity imply the latter: take $y' = 0$, then for any $\alpha \in [0,1]$ $\quad y \in Y \Rightarrow \alpha y + (1-\alpha) y' = \alpha y \in Y$.

b) Constant return to scale: $y \in Y \Rightarrow \alpha y \in Y \quad \forall \alpha \geq 0$.

Additivity: $y, y' \in Y \Rightarrow y + y' \in Y$

If $y, y' \in Y$, by convexity $\frac{1}{2} y + \frac{1}{2} y' \in Y$. Moreover, by constant return to scale, $2\left(\frac{1}{2} y + \frac{1}{2} y'\right) = y + y' \in Y$, thus it is additive.

c) Convex cone: $y, y' \in Y \Rightarrow \alpha y + \beta y' \in Y \quad \forall \alpha \geq 0, \beta \geq 0$

$y, y' \in Y \Rightarrow \alpha y, \beta y' \in Y$ by constant return to scale.

Then, by additivity $\alpha y + \beta y' \in Y$.

2) Assumptions: Y is closed, convex, satisfies free disposal.

Shows that it can be recovered from the profit function alone.

$\hat{Y} = \{ y \in \mathbb{R}^n : py \leq \pi(p) \text{ for all } p \in \mathbb{R}^n_+ \}$

Recall that $\quad \pi(p) = \max_y \; py \quad \text{s.t. } y \in Y$

(a) Let $y \in Y$. Then for any $p \in \mathbb{R}^n_+$, $py \leq \pi(p)$, by definition of $\hat{Y}$, we have that $y \in \hat{Y}$

Thus $Y \subset \hat{Y}$

(b) We want to show that $\hat{Y} \subset Y$. Let's prove the equivalent statement: $y \notin Y \Rightarrow y \notin \hat{Y}$.

Let $x \notin Y$ (we have to show that $x \notin \hat{Y}$)

Given that Y is convex and closed the seperating hyperplane theorem apply (Thm MG2 in the book). Thus there exists a strictly seperating hyperplane $H(p, a)$ such that

$py < a \quad \forall y \in Y$ and $px > a$

Assume for now that $p \in \mathbb{R}^n_+$. Then $\pi(p)$ is well defined and

$\forall y \in Y, \; py \leq \pi(p) = py^*$ for some $y^* \in Y$.

Given that $py^* < a$, it follows that $\pi(p) < a$

Therefore for our $x \notin Y$, $px > a > \pi(p) \Rightarrow px > \pi(p)$

Thus $x \notin \hat{Y}$ (it violates the definition for at least one p).

We complete the proof showing that $p \in \mathbb{R}^n_+$.

Suppose now that $p \notin \mathbb{R}^n_+$, i.e for at least a i, $p_i < 0$.

WLOG, $p_1 < 0$.

Then $p_1 y_1 + p_{-1} y_{-1} < a$

Given then free disposal, $\forall z_1 < y_1$, $(z_1, y_{-1}) \in Y$, thus

$p_1 z_1 + p_{-1} y_{-1} < a$

However we can take $z_1 < 0$ and $|z_1| \to +\infty$ and

$0 < pz_1 \to +\infty$. For z_1 large enough, $p_1 z_1 + p_{-1} y_{-1} > a$, which contradict the fact that $(z_1, y_{-1}) \in Y$.

Thus p_1 cannot be negative, and we can conclude that $p \in \mathbb{R}^n_+$.

Let $x \notin Y$ (we have to show that $x \notin \hat{Y}$)

Given that Y is convex and closed the seperating hyperplane theorem apply (thm M.G.2 in the book). Thus there exists a strictly seperating hyperplane $H(p,a)$ such that

$py < a \quad \forall y \in Y$ and $px > a$

Assume for now that $p \in \mathbb{R}^n_+$. Then $\pi(p)$ is well defined and

$\forall y \in Y, \; py \leq \pi(p) = py^*$ for some $y^* \in Y$.

Given that $py^* < a$, it follows that $\pi(p) < a$

Therefore for our $x \notin Y$, $px > a > \pi(p) \Rightarrow px > \pi(p)$.

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We complete the proof showing that $p \in \mathbb{R}^n_+$.

Suppose now that $p \notin \mathbb{R}^n_+$, i.e. for at least a i, $p_i < 0$.

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Given then free disposal, $\forall z_1 < y_1$, $(z_1, y_{-1}) \in Y$, thus

$p_1 z_1 + p_{-1} y_{-1} < a$.

However we can take $z_1 < 0$ and $|z_1| \to +\infty$ and

$0 < p z_1 \to +\infty$. For z_1 large enough, $p_1 z_1 + p_{-1} y_{-1} > a$,

which contradict the fact that $(z_1, y_{-1}) \in Y$.

Thus p_1 cannot be negative, and we can conclude that $p \in \mathbb{R}^n_+$.

3) $\boxed{f(z) = z_1 + z_2}$

(a) $\underbrace{(z_1, z_2, f(z_1,z_2))}_{\in Y} + \underbrace{(z'_1, z'_2, f(z'_1, z'_2))}_{\in Y} =$

$= (z_1 + z'_1, z_2 + z'_2, (z_1 + z_2) + (z'_1 + z'_2)) = \underbrace{(z_1 + z'_1, z_2 + z'_2, f(z_1 + z'_1, z_2 + z'_2))}_{\in Y}$

(b)

$$\min_{z \geq 0} wz \quad \text{s.t. } z_1 + z_2 \geq q \quad \rightarrow \quad \text{the value function is } c(w,q)$$

This is a linear objective function, with a linear constraint, thus,

$$z^*(w,q) = \begin{cases} (q,0) & \text{if } w_1 < w_2 \\ (0,q) & \text{if } w_1 > w_2 \\ \{(\lambda q, (1-\lambda) q) : \lambda \in [0,1]\} & \text{if } w_1 = w_2 \end{cases}$$

$$c(w,q) = w \cdot z^*(w,q)$$

$\boxed{g(z) = \min(z_1, z_2)}$

(a) $t = (z_1, z_2, \min(z_1, z_2)) \in Y$

$t' = (z'_1, z'_2, \min(z'_1, z'_2)) \in Y$

$(z_1 + z'_1, z_2 + z'_2, \min(z_1 + z'_1, z_2 + z'_2)) \in Y$

Does $t + t' \in Y$?

Notice that $\min(z_1, z_2) + \min(z'_1, z'_2) =$

$= \min\left(\underbrace{\min(z_1, z_2)}_{\leq z_1} + \underbrace{\min(z'_1, z'_2)}_{\leq z'_1}, \underbrace{\min(z_1, z_2)}_{\leq z_2} + \min(z'_1, z'_2) \right)$

In general $k = \min(k, k)$

$\leq \min(z_1 + z'_1, z_2 + z'_2)$

Thus, $t + t' \in Y$ (we implicitly ensured free disposal).

(b)

$$\min_{z \geq 0} wz \quad \text{s.t. } \min(z_1, z_2) \geq q$$

"Leontief" type.

$z^*(w,q) = (q,q)$

[By assumption $w >> 0$ otherwise it would be a correspondence]

$c(w,q) = (w_1 + w_2)q$

[$a_1, \dots, a_n$ $b_1, \dots, b_m \in \mathbb{R}^+$
$p < 1$ $(p \neq 0)$
$\left(\sum (a_k + b_k)^p\right)^{\frac{1}{p}} \geq \left(\sum a_k^p\right)^{\frac{1}{p}} + \left(\sum b_k^p\right)^{\frac{1}{p}}$]

$f(z) = \left(z_1^\rho + z_2^\rho\right)^{\frac{1}{\rho}}$ with $\rho < 1$

(a) Additivity follow directly from Minkowski inequality

(b)

$$\min wz \quad \text{s.t.} \quad \left(z_1^\rho + z_2^\rho\right)^{\frac{1}{\rho}} \geq q$$

FOC $\quad \lambda \frac{1}{\rho}\left(z_1^\rho + z_2^\rho\right)^{\frac{1}{\rho}-1} \rho \cdot z_i^{\rho-1} = w_i$

$$\Rightarrow \frac{z_1^{\rho-1}}{z_2^{\rho-1}} = \frac{w_1}{w_2} \quad \Rightarrow z_1 = \left(\frac{w_1}{w_2}\right)^{\frac{1}{\rho-1}} z_2$$

$$\Rightarrow \left(\left(\frac{w_1}{w_2}\right)^{\frac{\rho}{\rho-1}} + 1\right)^{\frac{1}{\rho}} z_2 = q$$

$$\Rightarrow \frac{\left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right)^{\frac{1}{\rho}}}{w_2^{\frac{1}{\rho-1}}} z_2 = q$$

$$\Rightarrow \begin{cases} z_2^* = \dfrac{q}{\left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right)^{\frac{1}{\rho}}} \cdot w_2^{\frac{1}{\rho-1}} \\[2ex] z_1^* = \dfrac{q}{\left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right)^{\frac{1}{\rho}}} \cdot w_1^{\frac{1}{\rho-1}} \end{cases}$$

$$c(w,q) = \frac{q}{\left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right)^{\frac{1}{\rho}}} \left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right) =$$

$$= q\left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}}$$

$z^*(w,q) = (q,q)$ [By assumption $w >> 0$ otherwise it would be a correspondence]

$c(w,q) = (w_1 + w_2) q$

[$a_1, \dots, a_n$ $b_1, \dots, b_n \in \mathbb{R}^+$
$p < 1$ $(p \neq 0)$
$\left(\sum (a_k + b_k)^p\right)^{\frac{1}{p}} \geq \left(\sum a_k^p\right)^{\frac{1}{p}} + \left(\sum b_k^p\right)^{\frac{1}{p}}$]

$h(z) = \left(z_1^\rho + z_2^\rho\right)^{\frac{1}{\rho}}$ with $\rho < 1$

(a) Additivity follow directly from Minkowski inequality

(b)
$$\min wz \quad \text{s.t.} \quad \left(z_1^\rho + z_2^\rho\right)^{\frac{1}{\rho}} \geq q$$

FOC $\lambda \frac{1}{\rho}\left(z_1^\rho + z_2^\rho\right)^{\frac{1}{\rho}-1} \rho \, z_i^{\rho-1} = w_i$

$$\Rightarrow \frac{z_1^{\rho-1}}{z_2^{\rho-1}} = \frac{w_1}{w_2} \Rightarrow z_1 = \left(\frac{w_1}{w_2}\right)^{\frac{1}{\rho-1}} z_2$$

$$\Rightarrow \left(\left(\frac{w_1}{w_2}\right)^{\frac{\rho}{\rho-1}} + 1\right)^{\frac{1}{\rho}} z_2 = q$$

$$\Rightarrow \frac{\left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right)^{\frac{1}{\rho}}}{w_2^{\frac{1}{\rho-1}}} z_2 = q$$

$$\Rightarrow \begin{cases} z_2^* = \dfrac{q}{\left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right)^{\frac{1}{\rho}}} \cdot w_2^{\frac{1}{\rho-1}} \\ z_1^* = \dfrac{q}{\left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right)^{\frac{1}{\rho}}} \cdot w_1^{\frac{1}{\rho-1}} \end{cases}$$

$$c(w,q) = \frac{q}{\left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right)^{\frac{1}{\rho}}} \left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right) =$$

$$= q \left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}}$$

4)

$c(w,q) \rightarrow$ cost in the long run

$c_s(w,q|z_1) \rightarrow$ cost in the short run

Goal: Show that $\frac{\partial q}{\partial p}(\bar{w},\bar{p}) \geq \frac{\partial q_s}{\partial p}(\bar{w},\bar{p})$

Let $\bar{q} = q(\bar{w},\bar{p})$ be the initial optimal amount at prices $\bar{w},\bar{p}$, and $q_s(\bar{w},\bar{p})$ the restricted one.

1 By assumption $c(\bar{w},\bar{q}) = c_s(\bar{w},\bar{q}|z_1)$

In general $f(q) = c(\bar{w},q) - c_s(\bar{w},q|z_1) \leq 0$, because costs can only increase when a constraint is added.

Thus $\bar{q}$ is a maximizer.

By regularity of the function the second derivative is negative:

$$\frac{\partial^2 f}{\partial q^2} = \frac{\partial^2 c(\bar{w},q)}{\partial q^2} \leq \frac{\partial^2 c_s(\bar{w},q|z_1)}{\partial q^2}$$

2 Consider the standard profit maximization:

$$\max_q \; pq - c(w,q)$$

FOC $\Rightarrow p = \left.\frac{\partial c(w,q)}{\partial q}\right|_{q=q^*}$

Similarly $p = \left.\frac{\partial c_s(w,q|z_1)}{\partial q}\right|_{q=q^*}$

We know $q^* = \bar{q} = q(w,p)$

Differentiating both side by p we obtain:

$$\frac{\partial^2 c}{\partial q^2} \cdot \frac{\partial q}{\partial p}(\bar{w},\bar{p}) = 1 = \frac{\partial^2 c_s}{\partial q^2} \frac{\partial q_s}{\partial p}(\bar{w},\bar{p})$$

We know that $\frac{\partial^2 c}{\partial q^2} \leq \frac{\partial^2 c_s}{\partial q^2}$ from 1, thus it should be

$$\frac{\partial q}{\partial p}(\bar{w},\bar{p}) \geq \frac{\partial q_s}{\partial p}(\bar{w},\bar{p})$$

PS 8

1) 5C8.

First, let's compute costs and profit.

<u>Month 3</u>
$c_1 = 3\times 40 + 1\times 50 = 170$
$\Pi_1 = 4\times 60 - 170 = 70$

<u>Month 95</u>
$c_2 = 2\times 55 + 2\times 40 = 190$
$\Pi_2 = 4\times 60 - 190 = 50$

However
$c'_2 = 2\times 40 + 2\times 50 = 180 < c_2$
and $\Pi'_2 = 4\times 60 - c'_2 = 60 > \Pi'_1$

$y = (-40, -50, 60) \in Y$, because it has been used in month 3, and it gives higher profit.

The firm does not seem to be profit maximiser (or there are additional constraints that we don't observe).

We cannot ensure that the observations are at the frontier and recover the boundaries of Y.

2) 5C13

The firm maximization problem is the following.

$$\max_{z_1, z_2} \; p f(z_1, z_2)$$
$$\text{s.t.} \quad w_1 z_1 + w_2 z_2 \leq C$$

given that f is concave we can ensure $=$ holds.

The proof will use envelope thm, we may directly apply theorem ML1, but we will do it step by step.

Assuming that f is (strictly) concave ensures that the maximisation problem has a solution and $z_1(p, w, c)$ and $z_2(p, w, c)$ are well defined functions.

Note that R is the value function of the maximisation problem, i.e.

$$R(p, w, c) = p f\big(z_1(p, w, c), z_2(p, w, c)\big)$$

We will show the following:

① $\frac{\partial R}{\partial w_1} = \lambda z_1(p, w, c)$ ② $\frac{\partial R}{\partial c} = -\lambda$

① and ② together allow us to conclude that

$$z_1(p,w,c) = -\frac{\partial R/\partial w_1}{\partial R/\partial c}$$ [Notice that this is basically Roy's identity.

$$= -\frac{(-\alpha/w_1)}{1/c} = \frac{\alpha c}{w_1}$$

① $$\frac{\partial R}{\partial w_1} = \frac{\partial}{\partial w_1}\Big(p\, f\big(z_1(p,w,c), z_2(p,w,c)\big)\Big) =$$

$$= p\frac{\partial f}{\partial z_1}\cdot\frac{\partial z_1}{\partial w_1} + p\frac{\partial f}{\partial z_2}\cdot\frac{\partial z_2}{\partial w_1} = (*)$$

[Now, recall that the solution of the constrained maximization problem is given by the following FOC

$$\begin{cases} p\dfrac{\partial f}{\partial z_1} = -\lambda w_1 \\ p\dfrac{\partial f}{\partial z_2} = -\lambda w_2 \end{cases}$$ → we can replace them in the above expression

$$(*) = -\lambda\left(w_1\frac{\partial z_1}{\partial w_1} + w_2\frac{\partial z_2}{\partial w_1}\right) = (**)$$

[Now recall that by construction

$$w_1 z_1(p,w,c) + w_2 z_2(p,w,c) = c$$
$$= g(z_1(p,w,c), z_2(p,w,c); w_1, w_2)$$

Thus $\frac{d g}{\partial w_1} = 0$, namely: $w_1\frac{\partial z_1}{\partial w_1} + w_2\frac{\partial z_2}{\partial w_1} + z_1 = 0$

$$\Leftrightarrow -z_1(p,w,c) = w_1\frac{\partial z_1}{\partial w_1} + w_2\frac{\partial z_2}{\partial w_2}$$

Replacing above, we obtain:

$$(**) = -\lambda\big(-z_1(p,w,c)\big) = \lambda z_1(p,w,c)$$

① and ② together allow us to conclude that

$$z_1(p,w,c) = -\frac{\partial R/\partial w_1}{\partial R/\partial c}$$ [Notice that this is basically Roy's identity.]

$$= -\frac{(-\alpha/w_1)}{1/c} = \frac{\alpha c}{w_1}$$

① $$\frac{\partial R}{\partial w_1} = \frac{\partial}{\partial w_1}\left(p f(z_1(p,w,c), z_2(p,w,c))\right) =$$

$$= p\frac{\partial f}{\partial z_1}\cdot\frac{\partial z_1}{\partial w_1} + p\frac{\partial f}{\partial z_2}\cdot\frac{\partial z_2}{\partial w_1} = (*)$$

[Now, recall that the solution of the constrained maximization problem is given by the following FOC

$$\begin{cases} p\dfrac{\partial f}{\partial z_1} = -\lambda w_1 \\ p\dfrac{\partial f}{\partial z_2} = -\lambda w_2 \end{cases}$$ → we can replace them in the above expression]

$$(*) = -\lambda\left(w_1\frac{\partial z_1}{\partial w_1} + w_2\frac{\partial z_2}{\partial w_1}\right) = (**)$$

[Now recall that by construction

$w_1 z_1(p,w,c) + w_2 z_2(p,w,c) = c$

$= g(z_1(p,w,c), z_2(p,w,c); w_1, w_2)$

Thus $\frac{dg}{dw_1} = 0$, namely: $w_1\frac{\partial z_1}{\partial w_1} + w_2\frac{\partial z_2}{\partial w_1} + z_1 = 0$

$\Leftrightarrow -z_1(p,w,c) = w_1\frac{\partial z_1}{\partial w_1} + w_2\frac{\partial z_2}{\partial w_2}$

Replacing above, we obtain:]

$$(**) = -\lambda(-z_1(p,w,c)) = \lambda z_1(p,w,c)$$

② $$\frac{\partial R}{\partial c} = \frac{\partial}{\partial c}\left(p f(z_1(p,w,c), z_2(p,w,c))\right) =$$

$$= p\frac{\partial f}{\partial z_1}\frac{\partial z_1}{\partial c} + p\frac{\partial f}{\partial z_2}\cdot\frac{\partial z_2}{\partial c} =$$ (same replacement as before)

$$= -\lambda\left(w_1\frac{\partial z_1}{\partial c} + w_2\frac{\partial z_2}{\partial c}\right) = (***)$$

[Now, notice that

$$\frac{d\,g(z_1(\cdot), z_2(\cdot); w_1, w_2)}{dc} = \frac{d}{dc}c = 1$$

i.e. $w_1\frac{\partial z_1}{\partial c} + w_2\frac{\partial z_2}{\partial c} = 1$. Replacing above:]

$$(***) = -\lambda$$

The demand function for the second good is found in an identical way:

$$z(p,w,c) = -\frac{\partial R/\partial w_2}{\partial R/\partial c} = \frac{(1-\alpha)c}{w_2}$$

③ 5G1

Let p be the price of output q, and let us normalize the price of the input at 1.

For a owner i: $u_i(z,q) = z + \tilde{u}_i(q)$

Moreover we can use the indirect utility function

$$v_i(p, w_i) = w_i + \varphi_i(p) \qquad (\text{see ex. } 3D4)$$

Here we can assume that the wealth come only from the profit (the solution would not be different if we allow for heterogeneous initial wealth)

(a) Each consumer would choose z to maximize the following:

$$\max_z \underbrace{\theta_i\left(p(z)f(z) - z\right)}_{\text{wealth in the share of profit.}} + \varphi_i\left(p(z)\right)$$

FOC: $\theta_i\left(p'(z)f(z) + p(z)f'(z) - 1\right) + \varphi_i'\left(p(z)\right)p'(z) = 0$

If owners agree on a z, this condition should be true for all of them.

$$\underbrace{\sum_i \theta_i}_{=1}\left(p'(z)f(z) + p(z)f'(z) - 1\right) = -\sum_i \varphi_i'\left(p(z)\right)p'(z)$$

[Recall that, by Roy identity $q_i(p) = -\dfrac{\partial v/\partial p}{\partial w/\partial w} = -\varphi_i'(p)$.
Thus, replacing in the above equality:

$$p'(z)f(z) + p(z)f'(z) - 1 = p'(z)\underbrace{\sum q_i(p)}_{= f(z) \to \text{the total production.}}$$

$$\Rightarrow \quad p(z)f'(z) = 1$$

Plugging this in the FOC, we find:

$$\theta_i\, p'(z)f(z) = q_i\left(p(z)\right)p'(z)$$

$$\Rightarrow \quad \theta_i = \frac{q_i\left(p(z)\right)}{f(z)}$$

(3) 5G1

Let p be the price of output q, and let us normalise the price of the input at 1.

For a owner i: $u_i(z, q) = z + \tilde{u}_i(q)$

Moreover we can use the indirect utility function

$$v_i(p, w_i) = w_i + \varphi_i(p) \qquad (\text{see ex. 3D4})$$

Here we can assume that the wealth come only from the profit (the solution would not be different if we allow for heterogeneous initial wealth)

a) Each consumer would choose z to maximize the following:

$$\max_z \; \underbrace{\theta_i \left(p(z) f(z) - z \right)}_{\text{wealth is the share of profit.}} + \varphi_i\left(p(z)\right)$$

FOC: $\theta_i \left(p'(z) f(z) + p(z) f'(z) - 1 \right) + \varphi_i'(p(z)) p'(z) = 0$

If owners agree on a z, this condition should be true for all of them.

$$\underbrace{\sum_i \theta_i}_{=1} \left(p'(z) f(z) + p(z) f'(z) - 1 \right) = - \sum_i \varphi_i'(p(z))\, p'(z)$$

[Recall that, by Roy identity $q_i(p) = - \dfrac{\partial v / \partial p}{\partial w / \partial w} = - \varphi_i'(p)$.
Thus, replacing in the above equality:

$$p'(z) f(z) + p(z) f'(z) - 1 = p'(z) \underbrace{\sum q_i(p)}_{= f(z) \to \text{ the total production.}}$$

$\Rightarrow \quad p(z) f'(z) = 1$

Plugging this in the FOC, we find:

$$\theta_i \, p'(z) f(z) = q_i(p(z))\, p'(z)$$

$$\Rightarrow \quad \theta_i = \frac{q_i(p(z))}{f(z)}$$

(b) From (a), we know that for consumers to agree on a production plan, then:

$$\theta_i = \frac{1}{N} = \frac{q_i(p(z))}{f(z)}$$

namely all the consumers should consume the same.
But this cannot be true if they have different preference for output and money.
More formally we can notice that $\theta_i = \theta_j$ implies

$\forall i, j \qquad \varphi_i'(p(z)) = \varphi_j'(p(z))$ for the optimal input.

this is in general false if $\varphi_i \neq \varphi_j$.

(c) Now the FOC are the same for everyone:

$$\frac{1}{N} \left(p'(z) f(z) + p(z) f'(z) - 1 \right) + \varphi'(p(z)) p'(z) = 0$$

and, as in (a):

$$p'(z) f(z) + p(z) f'(z) - 1 = p'(z) f(z)$$

$\Rightarrow \quad p(z) f'(z) = 1 \qquad p(z) = \dfrac{1}{f'(z)} \quad \to$ [$\varphi(p(z))$ does not appear here

Solving the profit maximisation give us:

$$\max_z \; p(z) f(z) - z$$

FOC $\quad p'(z) f(z) + p(z) f'(z) - 1 = 0$?

$$p(z) f'(z) = 1 - \underbrace{p'(z) f(z)}_{<0} > 1$$

$f'(z) \searrow$ in z, $p(z) \searrow$ in z

thus here we choose a smaller z.
This increases profit but not consumptions.
The owners-consumers prefer to be price-takers.
IF profit is maximized taking prices as given, then $p = \frac{1}{f'(z)}$ and the two FOCs are equivalent.