PS5 - MICRO

3.18 (a) Recall last problem set, the most general case are quasi-linear utility function.

In particular recall that we show that for i different from the numeraire (in this case the third good)

$$h_i(p,u) = h_i(p_1,p_2) = x_i(p_1,p_2) = x_i(p,w)$$

i.e. the hicksian demand for good i does not depend on utility and it is equivalent to the walrasian demand for good i (that in turns does not depend on wealth).

(b) Recall proposition 3G3: if demand is generated from preferences maximisation $S(p,w)$ is negative semidefinite, symmetric and satisfy $S(p,w)p = 0$ $\forall p$

Although it is perhaps longer, it may be easier to work without any normalisation on prices, having p_3 in the formula in order to satisfy walras law:

$$\begin{cases} x_1 = a_1 + b_1 \dfrac{p_1}{p_3} + c_1 \dfrac{p_2}{p_3} + d_1 \dfrac{p_1}{p_3}\cdot\dfrac{p_2}{p_3} \\ x_2 = a_2 + b_2 \dfrac{p_1}{p_3} + c_2 \dfrac{p_2}{p_3} + d_2 \dfrac{p_1}{p_3}\dfrac{p_2}{p_3} \\ x_3 = \dfrac{1}{p_3}(w - p_1x_1 - p_2x_2) \end{cases}$$

Let's start by focusing on the submatrix of the first two goods.

We take for now $p_3 = 1$ (if not, we would just have a $\frac{1}{p_3}$ multiplying everything)

For $i,j \in \{1,2\}$:

$$S_{ij} = \frac{\partial x_i}{\partial p_j} + \frac{\partial x_i}{\partial w}x_j = \frac{\partial x_i}{\partial p_j} \quad \text{because } \frac{\partial x_i}{\partial w} = 0$$

Thus $\quad s_{11} = b_1 + d_1p_2 \qquad s_{12} = c_1 + d_1p_1$

$\qquad\quad s_{21} = b_2 + d_2p_2 \qquad s_{22} = c_2 + d_2p_1$

- First, we show that $b_2 = c_1$.

Simmetry of S imposes: $b_2 + d_2p_2 = c_1 + d_1p_1 \quad \forall p_1, p_2$.

In particular, let $p_2 = \frac{d_1}{d_2}$, $p_1 = 1$.

Then $\quad b_2 + d_1 = c_1 + d_1 \Rightarrow b_2 = c_1$

Second, let's show that $d_1 = d_2 = 0$:

We know $c_1 + d_1 p_1 = c_1 + d_2 p_2$

$\Rightarrow \quad d_1 p_1 = d_2 p_2 \quad \forall p_1, p_2 \Rightarrow d_1 = d_2 = 0$

Simmetry impose also $s_{3i} = s_{i3}$.

Taking derivative we can see that this does not add restrictions.

E.g.: $$s_{13} = \frac{\partial x_1}{\partial p_3} = -\frac{1}{p_3^2}(b_1 p_1 + c_1 p_2)$$

$$s_{31} = \frac{\partial x_3}{\partial p_1} + \frac{\partial x_3}{\partial w} \cdot p_1 =$$

$$= -\frac{1}{p_3}\left(x_1 + \frac{\partial x_1}{\partial p_1} p_1 + \frac{\partial x_2}{\partial p_1} p_2\right) + \frac{1}{p_3} x_1 =$$

$$= -\frac{1}{p_3}\left(\frac{1}{p_3} b_1 p_1 + \frac{1}{p_3} c_1 p_2\right) = -\frac{1}{p_3^2}(b_1 p_1 + c_1 p_2) = s_{13}$$

- Let's now use the fact that the matrix is negative semidefinite.

This property is equivalent to the following fact:

[k^{th} order principal minors are non positive for k odd and non non negative for k even

Moreover we know that the rank of $S(p,w)$ is not maximum because $p^T S(p,w) p = 0$. Thus we know that the determinant of the matrix is 0; It follows:

1) $$\begin{cases} b_1 \leq 0 \\ c_2 \leq 0 \\ s_{33} \leq 0 \quad \Rightarrow 2 c_1 p_1 p_2 + c_2 p_2^2 + b_1 p_1^2 \leq 0 \\ \qquad\quad \Rightarrow (\text{for } p_1 = p_2 = 1) \quad c_1 \leq -\frac{1}{2}(c_2 + b_1) \end{cases}$$

2) $$b_1 c_2 - c_1^2 \geq 0$$

(I think other minors do not give additional conditions, I didn't check carefully)

3) $\det S(p,w) = 0 \rightarrow$ doing the computation one can see that it doesn't provide additional conditions.

- With the previous restrictions $S(p,w)\,p=0$ is always true.

(c) First, solve exercise 3I1 in MWG.

$p=(p_1,p_2,1) \quad | \quad p'=(p'_1,p'_2,1)$

$$EV(p,p',w) = e(p,u^*) - e(p,u') = e(p,u') - e(p',u') =$$

$$= e(p_1,p_2,u') - e(p'_1,p_2,u') + e(p'_1,p_2,u') - e(p'_1,p'_2,u') =$$

$$= \int_{p'_1}^{p_1} h_1(\pi_1,p_2,u')\,d\pi_1 + \int_{p'_2}^{p_2} h_2(p'_1,\pi_2,u')\,d\pi_2 =$$

$$\left[= EV((p_1,p_2),(p'_1,p_2),u) + EV((p'_1,p_2),(p'_1,p'_2),u) = \right]$$

$$= \int_{p'_1}^{p_1} x_1(\pi_1,p_2,w)\,d\pi_1 + \int_{p'_2}^{p_2} x_2(p'_1,\pi_2,w)\,d\pi_2 =$$

$$= \int_{p'_1}^{p_1} a_1 + b_1\pi_1 + c_1 p_2 \,d\pi_1 + \int_{p'_2}^{p_2} a_2 + c_1 p'_1 + c_2\pi_2 \,d\pi_2 =$$

$$= (a_1 + c_1 p_2)(p_1 - p'_1) + \frac{b_1}{2}(p_1^2 - p_1'^2) +$$

$$+ (a_2 + c_1 p'_1)(p_2 - p'_2) + \frac{c_2}{2}(p_2^2 - p_2'^2) \qquad (*)$$

We know (see discussion in the chapter) that for quasi-linear utility function if the price of the numeraire does not change, $EV = CV$, therefore we have a well defined measure of welfare.

(d) Let's replace the values in $(*)$ above. Take $b_1 = c_2 = -1$

(i) $EV_1 = 2(-1) - \frac{1}{2}(1-4) = -2 + \frac{3}{2} = -\frac{1}{2}$

(ii) $EV_2 = -\frac{1}{2}$

(iii) $EV_3 = -2 + \frac{3}{2} - \frac{5}{2} + \frac{3}{2} = -\frac{4}{2} + \frac{1}{2} = -\frac{3}{2}$

is always worse off when prices increase.

Obviously in this case $EV_3 \neq EV_1 + EV_2$.

In general, for the given price changes:

$$EV_1 = -(a_1 + c_1) - \frac{3}{2} b_1$$

$$EV_2 = -(a_2 + c_1) - \frac{3}{2} c_2$$

$$EV_3 = -(a_1 + c_1) - \frac{3}{2} b_1 - (a_2 + 2c_1) - \frac{3}{2} c_2$$

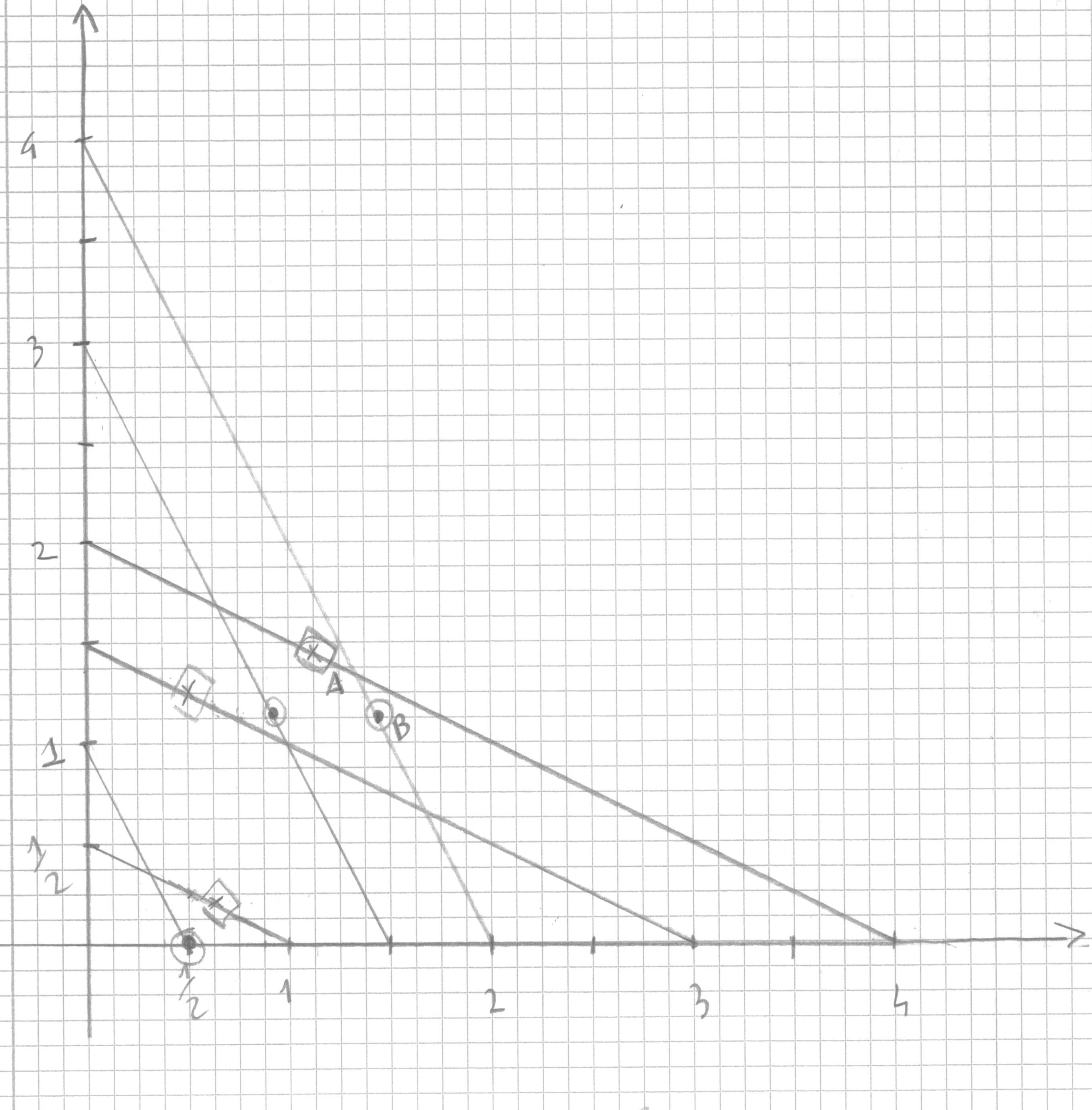
Thus $EV_1 + EV_2 = EV_3 \iff 2c_1 = 3c_1 \iff c_1 = 0$

In other words, there are no cross effect of prices.

4C12 Sketch of the solution

We build a counterexample. As suggested, two goods and two consumers, with wealth $w_1 = 1$, $w_2 = 3$. We pick prices $(1,2)$ and $(2,1)$. We ensure that the preferences at the two wealth level are as in the graph (WARP is not violated). However aggregating, WARP is violated.

[Note that $x(p,(w_1,w_2)) = x(p, F(\cdot))$ means that only the distribution of wealth matters, not the identity of the owners: consumers with the same wealth have identical demand
→ underlying preferences are the same].



(A) $p = (1,2)$ C_1: $x_1 = 0.625$ $x_2 = 0.1875$
C_2: $x_1 = 0.5$ $x_2 = 1.25$
$x_1 = 1.125$, $x_2 = 1.4375$

(B) $p = (2,1)$ C_1: $x_1 = 0.5$ $x_2 = 0$