PS 4 - Individual demand, part 2

) 3.C.6

(a) If $\rho = 1$ $u(x) = \alpha_1 x_1 + \alpha_2 x_2$

$\rightarrow$ Indifference curves are straight lines.

(b) The statement is true only if $\alpha_1 + \alpha_2 = 1$, thus we make this assumption. Moreover, let's ensure $\alpha_1 > 0, \alpha_2 > 0$

- First, notice that if one of the good is 0, both utilities are 0.

  WLOG, $x_2 = 0$, $x_1 > 0$

  CES: $u((x_1, 0)) = \alpha_1^{\frac{1}{\rho}} x_1 \xrightarrow[\rho \to 0]{} 0$ (because $\alpha_1 < 1$)

  C-D: $u'((x_1, 0)) = x_1^{\alpha_1} \cdot 0 = 0$

- Let consider the case $(x_1, x_2) \gg 0$

  For convenience, let's work with the monotonic transformation:

  $$\ln u(x) = \frac{1}{\rho} \ln \left(\alpha_1 x_1^{\rho} + \alpha_2 x_2^{\rho} \right)$$

  $$\lim_{\rho \to 0} \ln u(x) = \frac{\lim_{\rho \to 0} \ln(\alpha_1 x_1^{\rho} + \alpha_2 x_2^{\rho})}{\lim_{\rho \to 0} \rho} = \frac{\ln(\overbrace{\alpha_1 + \alpha_2}^{1})}{\lim_{\rho \to 0} \rho} \rightarrow \frac{0}{0}$$

  Using De L'Hopital rule, the above limit is equivalent to:

  $$\lim_{\rho \to 0} \ln u(x) = \lim_{\rho \to 0} \frac{\alpha_1 x_1^{\rho} \ln x_1 + \alpha_2 x_2^{\rho} \ln x_2}{\alpha_1 x_1^{\rho} + \alpha_2 x_2^{\rho}} = \frac{\alpha_1 \ln x_1 + \alpha_2 \ln x_2}{\underbrace{(\alpha_1 + \alpha_2)}_{=1}}$$

  $$= \alpha_1 \ln x_1 + \alpha_2 \ln x_2$$

  $\left[\text{Recall } \frac{d}{dt} a^t = a^t \ln a \right]$

  Now, applying a monotonic transformation, we can represent the utility using:

  $$\exp(\alpha_1 \ln x_1 + \alpha_2 \ln x_2) = x_1^{\alpha_1} x_2^{\alpha_2}$$, which is exactly the C-D.

We have to show that $\lim_{\rho \to -\infty} (\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)^{\frac{1}{\rho}} = \min(x_1, x_2)$

Let show that if $x_1 \leq x_2$, $\lim_{\rho \to -\infty} u(x) = x_1$.

The proof is identical if $x_2 \leq x_1$.

Recall "sandwich theorem": if $f(t) \leq g(t) \leq h(t)$ and $y = \lim_{t \to k} f(t) = \lim_{t \to k} h(t)$, then $\lim_{t \to k} g(t) = y$.

Given that $\rho \to -\infty$, we can safely take $\rho < 0$ in the proof. Then:

$x_1 \leq x_2 \iff x_1^\rho \geq x_2^\rho \iff (\alpha_1 + \alpha_2) x_1^\rho \geq \alpha_1 x_1^\rho + \alpha_2 x_2^\rho$

$\iff (\alpha_1 + \alpha_2)^{\frac{1}{\rho}} x_1 \leq (\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)^{\frac{1}{\rho}}$

Moreover $\alpha_1 x_1^\rho \leq \alpha_1 x_1^\rho + \underbrace{\alpha_2 x_2^\rho}_{\geq 0} \iff \alpha_1^{\frac{1}{\rho}} x_1 \geq (\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)^{\frac{1}{\rho}}$

Namely $(\alpha_1 + \alpha_2)^{\frac{1}{\rho}} x_1 \leq (\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)^{\frac{1}{\rho}} \leq \alpha_1^{\frac{1}{\rho}} x_1$

$$(\alpha_1 + \alpha_2)^{\frac{1}{\rho}} x_1 \xrightarrow{\rho \to -\infty} x_1 \qquad \alpha_1^{\frac{1}{\rho}} x_1 \xrightarrow{\rho \to -\infty} x_1$$

Thus $\lim_{\rho \to -\infty} (\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)^{\frac{1}{\rho}} = x_1$.

2) 3.E.7

- If $\succsim$ is quasilinear with respect to good 1, there exits a representation u such that $u(x_1, x_2, .., x_L) = x_1 + \varphi(x_2, .., x_L)$

- Let's show that $h(p,u) = x = (x_1, .., x_L) \Rightarrow h(p, u+\alpha) =$
$= (x_1+\alpha, x_2, ..., x_L)$
$= x + \alpha \cdot e_1$

This immediately implies the result: for any $u, u' \in \mathbb{R}$, there exits $\alpha \in \mathbb{R}$ such that $u' = u+\alpha$, therefore $\forall\, u, u' \in \mathbb{R}$,
$h(p,u') = h(p, u+\alpha) = x + \alpha \cdot e_1 = h(p,u) + \alpha \cdot e_1$

namely $\begin{cases} h_1(p,u') = h_1(p,u) + \alpha \\ h_i(p,u') = h_i(p,u) & \forall\, i \geq 2 \end{cases}$

We have to show that $(x_1+\alpha, x_2, .., x_L)$ solves the minimization problem

$$(\mathrm{I}) \quad \min_{z \geq 0} \; p \cdot z \quad \text{s.t. } u(z) \geq u+\alpha$$

First, notice that it satisfies the constraint:
$$u(x_1+\alpha, x_2, ..., x_L) = x_1 + \alpha + \varphi(x_2, .., x_L) = \overbrace{u(x_1, .., x_L)}^{\geq u} + \alpha \geq u+\alpha$$
because by assumption $x = (x_1, .., x_L)$ solves the minimization problem (II) $\min_{z \geq 0} p \cdot z$ s.t. $u(z) \geq u$

Second, let us show that for any other y that satisfies the constraint (i.e. $u(y) \geq u+\alpha$), $p \cdot y \geq p \cdot (x_1+\alpha, x_2, .., x_L)$, so that we can conclude that $(x_1+\alpha, .., x_L)$ minimizes the objective function

$u(y) \geq u+\alpha \iff y_1 + \varphi(y_2, .., y_L) \geq u+\alpha \iff$

$\iff y_1 - \alpha + \varphi(y_2, .., y_L) \geq u \iff u((y_1-\alpha, y_2, .., y_L)) \geq u$

Thus $(y_1-\alpha, .., y_L)$ satisfies the constraint in (II). Given that x is an optimal choice for (II), then

$p \cdot (y_1-\alpha, ..., y_L) \geq p \cdot x \iff \sum p_i y_i - p_1\alpha \geq \sum p_i x_i$

$\iff p \cdot y \geq p \cdot x + p_1 \alpha = p \cdot (x_1+\alpha, x_2, .., x_L)$

We can conclude that $(x_1+\alpha, x_2, ..., x_L)$ solves (I)

3) 3H3

Recall that $v(p,w) = u(x(p,w)) = u(x($

As stated in proposition 3E2, $e(p,u)$ is strictly
means that for any given p, the function
there exists $f_p(w): \mathbb{R}^+ \to \mathbb{R}$ s.t.

$$e(p, f_p(w)) = w$$

For $w = e(p,u)$, $u = u(x(p, e(p,u))) = v(p, e(p,u))$,

4) 3I6

Recall: compensating variation : $CV(p^0, p^1, w) = e(p^1, u^1) - e(p^1, u^0) = w - e(p^1, u^0)$

$$\sum_i CV_i(p^0, p^1, w_i) > 0 \iff \sum_i w_i > \sum_i e_i(p^1, u_i^0)$$

Let's define $w_i' = e_i(p^1, u_i^0)$. By construction $\sum_i w_i' \leq \sum_i w_i$,

moreover $v_i(p^1, w_i') = v_i(p^1, e_i(p^1, u_i^0)) = u_i^0 = v_i(p^0, w_i)$

(It is possible to redistribute resources so that everyone attains the previous utility).

3) 3H3

Recall that $v(p,w) = u(x(p,w)) = u(x(p, e(p,u)))$

As noted in proposition 3E2, $e(p,u)$ is strictly increasing in u. This means that for any given p, the function $e(p,\cdot)$ is invertible, i.e. there exists $f_p(w): \mathbb{R}^+ \to \mathbb{R}$ s.t. $f_p(e(p,u)) = u$ and

$e(p, f_p(w)) = w$

For $w = e(p,u)$, $u = u(x(p, e(p,u))) = v(p, e(p,u)) = v(p,w)$

4) 3I6

Recall: compensating variation: $CV(p^0, p^1, w) = e(p^1, u^1) - e(p^1, u^0) = w - e(p^1, u^0)$

$\sum_i CV_i(p^0, p^1, w_i) > 0 \iff \sum_i w_i > \sum_i e_i(p^1, u_i^0)$

Let's define $w_i' = e_i(p^1, u_i^0)$. By construction $\sum_i w_i' \leq \sum_i w_i$,

moreover $v_i(p^1, w_i') = v_i(p^1, e_i(p^1, u_i^0)) = u_i^0 = v_i(p^0, w_i)$

(It is possible to redistribute resources so that everyone attains the previous utility).