1. Consider two states of the world $\Omega = \{s_1, s_2\}$. An agent has preferences over state-contingent (monetary) payoffs $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2_+$. The price of consumption in state s_1 is normalized to 1 such that the market value of a bundle $\mathbf{x}$ is $x_1 + px_2$. Let the budget be w. The agent's preferences can be represented by a utility function $U(\mathbf{x})$.

   (a) Consider first the standard case in which an agent trusts her probability estimate $q = Pr(s_1) = 3/4$ and $(1-q) = Pr(s_2) = 1/4$. U satisfies expected utility and we have $U(\mathbf{x}) = q\log(x_1) + (1-q)\log(x_2)$. Show that the Walrasian demand function is $\mathbf{x}^*(p,w) = \frac{w}{4p}(3p, 1)$.

   Solution: Standard.

   (b) Depict the tangency-situation graphically, using that budget hyperplanes have normal vectors $(1, p)$ and preferred sets are supported by hyperplanes whose normal is $\nabla U(x_1, x_2) = (qu'(x_1), (1-q)u'(x_2))$.

   Solution: Standard picture.

   (c) From now on, we consider the case in which the agent does not have much confidence in her probability estimate of 1/2, and she considers instead a range of plausible $q \in [\frac{1}{4}, \frac{3}{4}]$. She wants to make consumption plans whose expected utility is fairly robust to the unknown probability q. More precisely, her utility function is

   $$V(\mathbf{x}) = \min_{q \in [\frac{1}{4}, \frac{3}{4}]} qu(x_1) + (1-q)u(x_2),$$

   with u increasing and concave. Find the minimizing probabilities as a function of $\mathbf{x}$:

   $$q^*(\mathbf{x}) = \arg \min_{q \in [\frac{1}{4}, \frac{3}{4}]} qu(x_1) + (1-q)u(x_2).$$

   Solution: The minimizing probability is the entire interval if $x_1 = x_2$, 3/4 if $x_2 > x_1$, and 1/4 if $x_1 > x_2$.

   (d) Show that a typical indifference curve has a kink on the 45-degree line. Use that the supporting hyperplanes have normal vectors $(q^*(\mathbf{x})u'(x_1), (1-q^*(\mathbf{x}))u'(x_2))$ at any point $\mathbf{x}$.

   Solution: The upper-contour set is the intersection between the two upper-contour sets with $q = 1/4$ and $q = 3/4$. There are infinitely many supporting hyperplanes at $x_1 = x_2$ for any normal vector which is a convex combination of $(0.25, 0.75)$ and $(0.75, 0.25)$.

   (e) Now assume $u(x) = \log x$. First, show that $p = 1$ implies $\mathbf{x}^*(1, w) = \frac{w}{2}(1, 1)$.

   Solution: In any of the answers below, just need to show that the price vector is collinear with the normal of the supporting hyperplanes and that Walras' law holds so that we can use the budget constraint as equality.

(f) Next, show that $p \in [\frac{1}{3}, 3]$ also implies full insurance $\mathbf{x}^*(p, w) = \frac{w}{1+p}(1, 1)$.

(g) Finally, show that $p < \frac{1}{3}$ implies $\mathbf{x}^*(p, w) = \frac{w}{4p}(3p, 1)$.

Solution: Here, we show that the consumer behaves as if $q = 3/4$.

2. Consider the family of vN-M functions

$$\mathcal{U} = \{u : [0, b] \to [0, b] | u(x) = \lambda \min\{x, 5\} + (1 - \lambda)x \text{ for some } \lambda \geq 0\}.$$

(a) Characterize the relation $\succsim_S$ on distribution functions with support in $[0, b]$ which satisfies that $F \succsim_S G$ if and only if $\int_0^b u(y)dF(y) \geq \int_0^b u(y)dG(y)$ for all $u \in \mathcal{U}$.

(b) Now consider the special case of F, G with $F(5) = G(5)$. Try to restate the condition you found in (a) in terms of (conditional) expectations.

(c) Is this stronger or weaker than FSD, SSD?

Solution: See Chapter I.3 in Gollier's textbook The Economics of Risk and Time. It suffices to consider the base functions $a(x) = \min\{x, 5\}$ and $b(x) = x$ of the family. Notice that $\int_0^b b(y)dF(y) \geq \int_a^b b(y)dG(y)$ if and only if $E_F[b(x)] = \int_0^b x dF(y) \geq E_G[b(x)]$. Integrating by parts, we get $\int_0^b a(y)dF(y) \geq \int_a^b a(y)dG(y)$ if and only if $\int_0^5 (G - F)(y)dy \geq 0$. That is, we can rank $F \succsim_S G$ if F has greater expected value than G and if the area under G exceeds the area under F up until 5. It is weaker, since both FSD and SSD satisfy both conditions. If $F(5) = G(5)$, then the condition is $F \succsim_S G$ if and only if $E_F(x|x \leq 5) \geq E_G(x|x \leq 5)$ and $E_F(x) \geq E_G(x)$.

Optional, but recommended (not required to hand in):

3. Solve 6.E.1 in MWG.

4. Find an example of two increasing (resp. increasing and concave) utility functions which disagree on the ranking of two distributions F, G that cannot be ranked by FSD (resp. by SSD).

5. Consider two probability density functions f and g, strictly positive on their common support $[a, b]$. We say that f dominates g according to the *monotone-likelihood-ratio* order (MLR) if, for all $x, y \in [a, b]$, we have $f(x)/g(x) \geq f(y)/g(y)$ whenever $x \geq y$. Show that MLR is a special case of FSD. *Hint*: Rearrange, then integrate the inequality twice.

Solution: In fact, there is a complete proof on Wikipedia.