MICRO I – PS 1

EX1. Recall the following definitions:

① Completness: $\forall x, y \in X$, $x \succsim y$ or $y \succsim x$
② Transitivity: $\forall x, y, z \in X$, $x \succsim y$ and $y \succsim z \Rightarrow x \succsim z$

Recall that from $\succsim$ we derive $\succ$ and $\sim$

1. By completness, applying the definition with $y = x$,
$x \succsim x$
By definition of $\succ$, $x \succ y \Leftrightarrow x \succsim y$ and not $y \succsim x$
For $y = x$, $x \succ x \Leftrightarrow x \succsim x$ and not $x \succsim x$
We know $x \succsim x$ is true, thus not $(x \succsim x)$ is false and the entire condition is false. Thus $\succ$ is irreflexive.

2. By completness, $x \succsim x$.
By def of $\sim$, $x \sim y \Leftrightarrow x \succsim y$ and $y \succsim x$.
For $y = x$, we conclude $x \sim x$

3. Note that by def of $\succ$, $x \succ y \Rightarrow x \succsim y$.
Given that $x \succsim y$ and $y \succsim z$, by transitivity $x \succsim z$.
Suppose (by contraddiction) that $z \succsim x$. Then, by transitivity (given that $y \succsim z$ and $z \succsim x$) $y \succsim x$, which contraddicts the initial assumption that $x \succ y$.
Having reached the desired contraddiction, we can conclude that not $(z \succsim x)$. Given that $x \succsim z$, by definition of $\succ$, we can conclude that $x \succ z$.

EX3. a. The set of alternatives is
$X = B \times G = \{(b, g) : b \in B, g \in G\} = \{(b_1, g_1), (b_1, g_2), (b_2, g_1), (b_2, g_2), (b_3, g_1), (b_3, g_2)\}$

b. $A = \{ \{(\beta_1, g_1), (\beta_2, g_2)\} \mid \beta_1 \in B, \beta_2 \in B, \beta_1 \neq \beta_2 \} =$
$= \{ \{(b_1, g_1), (b_2, g_2)\}, \{(b_1, g_1), (b_3, g_2)\}, \{(b_2, g_1), (b_1, g_2)\},$
$\{(b_2, g_1), (b_3, g_2)\}, \{(b_3, g_1), (b_2, g_2)\}, \{(b_3, g_1), (b_2, g_2)\}\}$

c. Yes, the choice rule consists in picking the couple with g_2

$$C(\{(b_1,g_1),(b_2,g_2)\}) = C(\{(b_3,g_1),(b_2,g_2)\}) = \{(b_2,g_2)\}$$

$$C(\{(b_2,g_1),(b_1,g_2)\}) = C(\{(b_3,g_1),(b_1,g_2)\}) = \{(b_1,g_2)\}$$

$$C(\{(b_1,g_1),(b_3,g_1)\}) = C(\{(b_2,g_1),(b_3,g_2)\}) = \{(b_3,g_2)\}$$

Note that with a different $\mathcal{A}$ it wouldn't be a well defined choice rule.

d. Yes. If $x \succ_C y$, then $\exists A \in \mathcal{A}$, s.t. $x \in A$ and $y \in A$.
Then, by def of $\mathcal{A}$, $x = (\beta_1, g_2)$ for some $\beta_1 \in B$ and
$y = (\beta_2, g_1)$ for some $\beta_2 \in B$, $\beta_1 \neq \beta_2$
Then, by definition of C, $\nexists D \in \mathcal{A}$, s.t. $y \in C(D)$. The WARP axiom is "trivially" verified

Ex. 4. No. $c \succ_C a$ because $a, c \in \{a,b,c\}$ and $c \in C(\{a,b,c\})$
Given that $a, c \in \{a,c,d\}$ and $a \in C(\{a,c,d\})$,
WARP is satisfied if $c \in C(\{a,c,d\})$.
However by def. $c \notin C(\{a,c,d\}) = \{a\}$.
Thus WARP is violated.

EX 2.

(A) First, let's show that if $\succsim$ is a rational preference, then $\succ$ satisfies Kreps axiom.

Asimmetry: Let $x, y \in X$. By completeness, either $x \succsim y$ or $y \succsim x$. WLOG, suppose $x \succsim y$.

- If $y \succsim x$, then $x \sim y$ and not$(x \succ y)$, not$(y \succ x)$, thus the statement is satisfied

- If not$(y \succsim x)$, then by def of $\succ$, $x \succ y$ and not$(y \succ x)$. Again, the statement is satisfied.

Negative transitivity: Let $x, y, z \in Z$ and $x \succ y$.

By completeness $y \succsim z$ or $z \succsim y$. ① If $y \succsim z$, then $x \succ y \succsim z \Rightarrow x \succ z$ (by Ex. 1, 3).

② if $z \succsim y$ and not$(y \succsim z)$, $z \succ y$ by def of $\succ$

(B) Second, let's show that if the binary relation P satisfies Kreps axioms, then there exists $\succsim$ such that ① P is the strict preference relation associated with $\succsim$, ② $\succsim$ is rational.

We can "extend" P in order to get the following preference relation:

$\succsim = P \cup \{(x, y) \in X : \text{not}(xPy) \text{ and not}(yPx)\}$

In other words $x \succsim y \iff xPy$ or $((x,y) \notin P$ and $(y,x) \notin P)$.

① - Let's show that $x \succ y \iff xPy$.

$\Rightarrow$) $x \succ y \iff$ (by def of $\succ$) $x \succsim y$ and not$(y \succsim x)$.

Suppose (by contraddiction) not(xPy). It cannot be not(yPx), otherwise $(x,y) \notin P$, $(y,x) \notin P \Rightarrow (x,y) \in \succsim$, $(y,x) \in \succsim$

$\Rightarrow (y \succsim x)$, contraddicting the initial assumption

Thus yPx, but then $y \succsim x$ contraddicting again the assumption.

We can conclude that xPy.

$\Leftarrow$) $xPy \Rightarrow x \succsim y$. Moreover, given that P satisfies asimmetry, not(yPx). Given that xPy and $(x,y) \in P$, we can conclude that $(y,x) \notin \succsim$ i.e. not$(y \succsim x)$.

Thus $x \succ y$.

2) Let's show that $\succsim_1$ is complete and transitive

<u>Completeness.</u> $\forall x, y \in X$, either xPy or not(xPy)

If xPy, then $x \succsim_1 y$, in particular $x \succ y$

If not(xPy), then either yPx or not(yPx)

If yPx, then $y \succsim_1 x$ (in particular $y \succ x$)

If not(yPx), then $(x,y) \notin P$ & $(y,x) \notin P$ $\rightarrow$

(by def. of $\succsim_1$) $(x,y) \in \succsim_1$, i.e. $x \succsim_1 y$ (in particular $x \sim y$)

<u>Transitivity</u> Suppose $x \succsim_1 y$ and $y \succsim_1 z$. We want to show that $x \succsim_1 z$.

1) Suppose $x \succ y$, i.e. xPy. Then by negative transitivity either xPz or zPy. However, given that $y \succsim_1 z$, it can't be $z \succ y$. Thus xPz, which implies $x \succsim_1 z$.

2) Suppose not$(x \succ y)$ i.e. $y \succsim_1 x$. $(x \sim y)$.
Suppose yPz. Then by neg. trans., either yPx or xPz
Given that it can't be $y \succ x$, it should be $x \succ z$,
i.e. $x \succsim_1 z$

3) Suppose $x \sim y$ and $y \sim z$.
If xPz, then, by neg trans., either $x \succ y$ or $y \succ z$. It can't be.
If zPx, then $z \succ y$ or $y \succ y$. It can't be.
Thus, by def. of $\succsim_1$, $(x,z) \in \succsim_1$, $x \succsim_1 z$