Ex-ante Efficiency in Assignments with Seniority Rights

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Abstract

We study random assignment economies with expected-utility agents, each of them eventually obtaining a single object. We focus attention on assignment problems that must respect object-invariant (or uniform) weak priorities such as seniority rights in student residence assignment. We propose the *Sequential Pseudomarket* mechanism: The set of agents is partitioned into ordered priority groups that are called in turns to participate in a pseudomarket for the remaining objects. SP is characterized by the concept of *Consistent Weak Ex-ante Efficiency* (CWEE), that is, Weak Ex–ante Efficiency complemented by Consistency to economy reduction. Moreover, it is shown that CWEE *generically implies Ex-ante Efficiency.*

Keywords: Random assignment, ex-ante efficiency, consistency, sequential pseudomarket

JEL codes: D47, D50, D60

1 Introduction

In a random assignment, each agent is provided with a probability distribution over the set of object types. Agents have preferences over their assigned distributions according to the expected utility form. No monetary transfers are allowed. Hylland and Zeckhauser's (1979) seminal paper suggests that a pseudomarket can be constructed in which each agent is endowed by some artificial income with which she can buy assignment probabilities. Each object type is given a nonnegative price and each agent buys a proper probability distribution (probabilities add up to 1) over them. Given the endowment vector, there is at least one equilibrium price vector yielding a feasible random assignment as an outcome. Moreover,

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this random assignment is ex-ante Pareto-efficient, in a sort of First Theorem of Welfare Economics for random assignment economies. Miralles and Pycia (2014) showed the Second Welfare Theorem counterpart.

However, preexisting priority rights are found in many assignment problems. For example, in school choice, a child whose parents apply for the last slot at a public school cannot typically occupy it if the parents of another child with a sibling already attending the school want that slot (the so-called sibling priority). There are many priority criteria in many different assignment problems: proximity to the school, low income, or being organ donor in "kidney exchange".

This paper focuses attention on (possibly weak) priority structures that are object-invariant (uniform), that is, independent from the object for which agents are competing. A paradigmatic example of this kind of problems is the assignment of students to college residences with seniority rights. *The practical motivation question of this paper is whether there is a mechanism that respects object-invariant priorities while it attains good ex-ante efficiency properties.* Is the respect for uniform priorities compatible with ex-ante Pareto-efficiency?

As we will see, the answer is "generically yes", and a simple mechanism is proposed that achieves both objectives. We introduce the **Sequential Pseudomarket (SP)**. In SP, ordered groups of agents (top-priority agents, second-priority agents...) are called in turns that participate in the pseudomarket for the remaining objects. A SP-equilibrium is a sequence of pseudomarket equilibria turn by turn. It is easy to see that SP encompasses a family of mechanisms whose opposite extremes are serial dictatorship and pseudomarkets without priorities.[1] Considering ordered groups as priority groups, it is also straightforward to see (Lemma 1) that any SP-equilibrium assignment respects uniform priorities in the ex-ante stability sense (Kesten and Ünver, 2015). This makes SP suitable for random assignment problems with uniform weak priorities.[2]

In principle, the SP mechanism cannot guarantee that its equilibrium outcome is ex-ante efficient.[3] For that reason we propose a new, weaker notion of efficiency, namely **Consistent Weak Ex-ante Efficiency (CWEE)**. This notion of efficiency is the result of applying the Consistency requirement (see Thomson, 2015, for a recent survey on this concept and its relevance) to the notion of Weak Ex-

[1]Informally, one extreme is the set of allocations rendered by the finest ordered partitions (strict uniform priorities), while the other extreme compresses random assignments resulting from the singleton partition (no priorities.)

[2]In a very recent paper, Han (2015) also studies random assignments with priority groups ordered hierarchically. He focuses on ordinal preferences, hence he designs generalizations of Serial Dictatorship and Probabilistic Serial to these priority structures. While the former mechanism is designed to guarantee ex-post efficiency (no mutually beneficial exchange of final allocations) and the latter aims at the finer notion of ordinal efficiency (no first-order stochastically dominating feasible redistribution of probabilities), the SP mechanism suggested in this paper generically satisfies the even *finer notion* of ex-ante efficiency (no mutually beneficial trade of assigned probabilities).

[3]A random assignment is ex-ante efficient if there is no feasible redistribution of probabilities in which everyone is ex-ante weakly better-off, with at least one agent being strictly ex-ante better-off. Example 1 in the main text illustrates that SP does not guarantee this property.

ante Pareto-efficiency of a random assignment.[4] Consistency in this context means that, after removing *any* set of individuals and their *assigned probabilities* from the economy, the weak ex-ante efficiency property of the random assignment *holds in the remaining economy.*[5] In a characterization result, we show that *a random assignment is CWEE if and only if it can be generated by an SP-equilibrium* for some partition of the set of agents into ordered groups (Theorem 1). This result contains both First and a Second Welfare Theorems for random assignment economies, when the efficiency notion is *CWEE.*[6]

It is easy to see that Ex-ante Efficiency implies CWEE, which in turns implies Weak Ex-ante Efficiency. Converses are not true in the random assignment economies we study. To illustrate that the latter property does not imply any of the formers, one could construct an easy example with three agents x, y and z and three unit-supplied objects a, b and c. All agents' favorite object is a. Agent y prefers b to c while agent z prefers c to b. We assign object a to x, and objects b and c evenly to y and z. Such an assignment is weakly ex-ante efficient, since no reassignment would strictly improve agent x's assignment. However it is not CWEE, since the remaining economy with objects b and c and agents y and z has a strictly Pareto-improving reallocation: the sure assignment of b to y and c to z. A more elaborated example (Example 1 in the main text) would illustrate that CWEE does not forcefully imply ex-ante efficiency. But then, how far is CWEE from ex-ante efficiency?

Theorem 2 brings good news: *we can generically state that every CWEE (and hence any SP-equilibrium) random assignment is ex-ante Pareto-optimal.*[7] This result is somewhat striking since differently priority-ranked agents face different relative prices, a fact that could have caused inefficiency on the random assignment. A second look at the problem clarifies it. Theorem 1 allows us to think of CWEE random assignments as SP-equilibrium random assignments. In a SP-equilibrium random assignment, no agent could be strictly better-off after trading assignment

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[4] A random assignment is weakly ex-ante efficient if there is no other feasible random assignment in which all agents in the economy are strictly ex-ante better-off.

[5] Chambers (2004) suggests a more restrictive notion of probabilistic consistency by which each agent leaving the economy realizes a draw from her assignment probabilities and leaves with a sure object. Instead, our approach uses the argument in Thomson (2015, page 215): "An alternative approach is to think that, to begin with, each object is available with a certain probability that is not necessarily equal to 1. [...] When an agent leaves with his assignment, namely a vector of probabilities of receiving the various objects, the probability of each object being available to the remaining agents is decreased by the probability with which it has been assigned to the agent who leaves."

[6] CWEE is also related to Roth and Postlewaite's (1977) notion of strong-domination stability. A final allocation is strong-domination stable if it is in the weak core of a market where the final allocation is taken as the endowment. Our notion constitutes the extension of their notion to random assignments. However, we prefer to understand CWEE as an efficiency concept rather than a stability concept, since core concepts are more easily understood as related to initial endowments.

[7] To be precise, the set of preference profiles under which there is equivalence between CWEE and ex-ante efficiency is dense in the space of preference profiles.

probabilities with lower-ranked agents. Whatever the latter agents obtained, it was zero-priced for the former agent, and she discarded it. However, it is still possible that some trades leave the former agent indifferent while benefitting lower-ranked agents.

Two generically met assumptions disregard the latter concern. First, that no agent is indifferent between two object types (Assumption 1.)[8] Second, a *regularity condition for preferences* (Assumption 2,) which embeds the assumption of Bonnisseau, Florig and Jofré (2001) used for linear utility economies (see their Lemma 4.1,) namely that there is no cycle of marginal rates of substitution of different agents that multiplied altogether yield one. Our assumption is quite technical yet it can be explained from the next question: From each possible initial assignment, is there any feasible redistribution of probabilities between two or more agents in which *all* of the affected agents remain indifferent? If the answer is no for all possible initial random assignments (which happens generically[9]), the regularity condition is satisfied. Our assumptions 1 and 2 imply an important side result (Proposition 1): each pseudomarket price equilibrium has a unique associated equilibrium assignment. And ex-ante suboptimality of a CWEE allocation (the concern in the previous paragraph) can only arise when this is not the case, as shown along the proof of Theorem 2.

We remark at this point that Weak Ex-ante Efficiency alone does not generically imply Ex-ante Efficiency. The generic property of CWEE in Theorem 2 is not the fruit of a "sandwiching effect" between Weak Ex-ante Efficiency and Ex-ante Efficiency.

More than thirty years after the seminal paper by Hylland and Zeckhauser, pseudomarkets are attracting increasing interest both in finite and continuum economies.[10] Examples of recent papers are Azevedo and Budish (2015) on strategy-proofness in the large that applies to pseudomarkets, or Budish, Che, Kojima and Milgrom (2012) on pseudomarket mechanisms for multidimensional assignment. We contribute to this literature by providing a proper and simple combination between pseudomarket and serial dictatorship that performs satisfactorily in assignment problems with object-invariant priority structures.

This paper is closely related with two other recent pieces of research: Miralles and Pycia (2014) and He, Miralles, Pycia and Yan (2015). The first paper establishes a Second Welfare Theorem for random assignment economies. In virtue of this result, one could have obtained any ex-ante Pareto-efficient assignment, including one that respects uniform priorities, by fine-tuning individual incomes and then letting agents purchase probability bundles in a competitive market. The practical advantage of the approach taken in the current paper is that, instead of adapting incomes to agents' preferences so that uniform priorities are respected,

[8] This assumption can be rapidly side-stepped by adding some additional structure on pseudomarket allocations. For instance, Hylland and Zeckhauser (1979) impose that each agent, when being indifferent among any two bundles, chooses the least expensive one. In this way, ex-ante Pareto-optimality of any Pseudomarket allocation is guaranteed.

[9] Remark 2 in the main text serves to notice that the answer to the previous question is yes only when some matrix of indifference-holding vectors of probability redistributions is singular.

[10] See Thomson and Zhou (1993) and Ashlagi and Shi (2015) for a result on efficient and fair allocations in continuum economies.

we just need to assign turns. Success in reaching an Ex-ante Efficient random assignment is generically guaranteed. The SP method becomes informationally less demanding, and hence easier to implement.

The second paper analyses the question of how we can adapt the Pseudomarket mechanism to meet any set of (possibly weak) priority criteria. This includes uniform priorities as a special case. The suggested solution is an alteration of prices depending on priority status. For each object type there would be a critical priority level which pays the market price. Instead, higher priority levels enjoy zero price for the object, whereas lower priority levels face infinite price. The Sequential Pseudomarket is an example of such a mechanism for the case of uniform priorities. Sequential Pseudomarkets deserve however particular attention, since they generically guarantee agent-side ex-ante efficiency, a nice property when the other side of the market is constituted by objects. For more general priorities, only two-sided unconstrained efficiency is guaranteed, considering priorities as objects' (weak) ordinal preferences.

Section 2 presents the basic notation and definitions of the model. Section 3 introduces SP and its stability properties. Section 4 contains the Welfare Theorems linking SP with CWEE. Section 5 establishes generic equivalence between CWEE and Ex-ante Efficiency. Section 6 concludes. An appendix contains the proof of Proposition 1 and additional analysis regarding Assumption 2.

2 The model: notation and definitions

Random assignment and preferences

There is a finite set of agents $N = \{1, ..., n\}$. The notation x or y is used for a generic element of N. There is a set of object types $S = \{1, ..., s\}$, with $s \geq 3$. The notation i or j serves to indicate a generic element of S. For each object type j there is a number of identical copies $\eta^j \in \mathbb{N}$. $\eta = (\eta^1, ..., \eta^s)$ is the supply in this economy. We have enough supply in the sense that $\sum_{j \in S} \eta^j \geq n$.[11]

A random assignment is a $n \times s$ matrix Q whose generic element $q_x^j \geq 0$ is the probability that agent x obtains a copy of object type j. This matrix is stochastic: $\sum_{j \in S} q_x^j = 1$ for any $x \in N$. Agent x's random assignment is the probability distribution $q_x = (q_x^1, ..., q_x^s) \in \Delta^s$ (Δ^s is the $s-1$ dimensional simplex). A random assignment is a pure assignment if each of its elements is either 1 or 0. A random assignment is feasible if $Q^T \cdot 1_n \leq \eta$ (where 1_n is a vector of n ones and ^{T} denotes the transpose of a matrix). A feasible random assignment can be expressed as a lottery over feasible pure assignments.

Let $V \in \mathbb{R}_+^{n \times s}$ denote a $n \times s$ matrix of nonnegative von Neumann-Morgenstern valuations, whose generic element v_x^j indicates agent x's valuation for object type j. A generic agent x's valuation vector is $v_x = (v_x^1, ..., v_x^s)$. She values her random assignment q_x as the vectorial product $u_x(q_x) = v_x \cdot q_x$. Each agent x has a set of most-preferred object types $M_x = \arg\max_{j \in S} v_x^j$. An agent x is *satiated* if q_x contains positive probabilities for objects in M_x only. An economy is a triple $E = (N, \eta, V)$.

Efficiency notions

[11] Notice that the weak inequality allows for an easy inclusion of an outside option for every agent.

Let $\mathcal{F}_E$ denote the set of feasible random assignments in an economy E. A feasible random assignment Q^* is *ex-ante Pareto-optimal* (or ex-ante efficient) at an economy E if for any random assignment Q, $q_x \cdot v_x \geq q_x^* \cdot v_x$ $\forall x \in N$ (with strict inequality for some x)$\Longrightarrow Q \notin \mathcal{F}_E$.

For a feasible random assignment Q^*, let a *trading coalition* $C \subset N$ from Q^* be defined as follows: $\exists Q$ such that a) $q_x^* \cdot v_x < q_x \cdot v_x$ for all $x \in C$, b) $q_x^* = q_x$ for all $x \in N \backslash C$, and c) $\sum_{x \in C} q_x \leq \eta - \sum_{x \in N \backslash C} q_x^*$. A feasible random assignment Q^* is *Consistent Weak Ex-ante Efficient (CWEE)* for an economy E if it admits no nonempty trading coalition C. A feasible random assignment Q^* is *weakly ex-ante efficient* for an economy E if N is not a trading coalition from Q^*.

Prices and equilibrium

A price vector is notated as $P \in \mathbb{R}_+^s$. A price vector P^* constitutes a pseudomarket *quasiequilibrium* for an economy E with associated feasible random assignment Q^* if for any random assignment Q and any agent x we have $u_x(q_x) > u_x(q_x^*) \Longrightarrow P^* \cdot q_x \geq P^* \cdot q_x^*$. A price vector P^* constitutes a pseudomarket *equilibrium* for an economy E with associated feasible random assignment Q^* if for any random assignment Q and any agent x we have $u_x(q_x) > u_x(q_x^*) \Longrightarrow P^* \cdot q_x > P^* \cdot q_x^*$. We restrict attention to equilibria satisfying the slackness condition: $\sum_{x \in N} q_x^{*i} < \eta^i$ implies $P^{*i} = 0$. Existence of equilibria satisfying this condition is proven, for instance, in He, Miralles, Pycia and Yan (2015).

We have not explicitly modeled individual budget limits in the preceding definitions. They are implicitly defined though, being equal to $P^* \cdot q_x^*$ if agent x is not satiated, or not lower than $P^* \cdot q_x^*$ if agent x is satiated. Our notation implicitly accommodates from simple budget distributions (e.g. the usual equal income for all agents) to richer, history-dependent budgets.

Priorities and stability

A *priority* for object type $j \in S$ is a weak linear ordering $\succsim^j$ over the elements in N. $\succ^j$denotes the strict part of $\succsim^j$ while $\sim^j$ denotes the indifference part of $\succsim^j$. A priority structure is a profile $(\succsim^j)_{j \in S}$.

A priority is a rule to resolve conflicting demands: if two agents claim for the unique remaining copy of an object type, the agent with higher priority must obtain it. This definition admits weak priorities: both x and y could be considered at the same priority level for some object type j.

A feasible random assignment Q is *ex-ante stable* (or it respects priorities ex ante) à la Kesten and Ünver (2015) if for any object $j \in S$ and any two agents $x, y \in N$ we have that $x \succ^j y$ and $q_y^j > 0$ imply $q_x^i = 0$ for all objects $i \in S$ such that $v_x^i < v_x^j$.

Uniform priorities arise when each agent x has priority over agent y for object i if and only if x has priority over agent y for every other object. Formally, a priority structure $(\succsim^j)_{j \in S}$ is *object-invariant* (or uniform) if for every $i, j \in S$ we have $\succsim^i = \succsim^j$. An object-invariant priority structure gives rise to an *ordered partition* of N into a collection of disjoint sets $N_1, ..., N_\pi$, such that for every $t < \tau$, $x \in N_t$ and $y \in N_\tau$, we have that $x \succ^j y$ for all $j \in S$. We also say in this case that the ordered partition $N_1, ..., N_\pi$ *induces uniform priorities* $(\succsim^j)_{j \in S}$ if for every $t < \tau$, $x \in N_t$ and $y \in N_\tau$, we have that $x \succ^j y$ for all $j \in S$. For the rest of the paper we focus on models with object-invariant priorities.

Provided an ordered partition $N_1, ..., N_\pi$, we say that a random assignment Q *respects uniform priorities induced by* $N_1, ..., N_\pi$ in the Kesten-Ünver ex-ante stability sense (Kesten and Ünver, 2015) if this condition holds: if for $t \in \{2, ..., \pi\}$ and for some $i \in S$ we have $\sum_{x \in N_t} q_x^{i*} > 0$, then for all $x \in N_1 \cup ... \cup N_{t-1}$ it must be the case that for every $j \in S$, $v_x^i > v_x^j \Longrightarrow q_x^{j*} = 0$.[12]

3 Sequential pseudomarkets and ordered priority sets

We first introduce the key idea of this paper, the presentation of an intuitive mechanism that works well under object-invariant priorities.

Definition 1 *(SP Mechanism and Equilibrium) The SP mechanism with ordered partition $N_1, ..., N_\pi$ at economy E proceeds as follows. First, the set N is partitioned into disjoint ordered sets $N_1, ..., N_\pi$ with $\pi \leq n$. We start with a reduced economy E_1 with N_1 on the demand side and $\eta_1 = \eta$ as the supply side. Calculate a pseudomarket equilibrium price vector P_1^* jointly with an associated random allocation Q_1^* for this reduced economy. For $t = 2, .., s$, calculate the remaining supply $\eta_t = \eta_{t-1} - Q_{t-1}^{*\prime} \cdot 1_{|N_{t-1}|}$ and use N_t on the demand side to calculate a new pseudomarket equilibrium price vector P_t^* with an associated random allocation Q_t^* for the reduced economy $E_t = (N_t, \eta_t, V_t)$.[13] (V_t is a selection from V that contains the preferences for agents in N_t)*

The array of price vectors $[P_1^*, ..., P_\pi^*]$ constitutes a Sequential Pseudomarket (SP)- equilibrium price matrix at economy E given the ordered partition $N_1, ..., N_\pi$. The vertical composite matrix $Q^* = [Q_1^*, ..., Q_\pi^*]$ is a SP-equilibrium random assignment associated to $[P_1^*, ..., P_\pi^*]$ at economy E given the ordered partition $N_1, ..., N_\pi$.[14]

Remark 1 *When $\pi = n$ SP becomes a Serial Dictatorship. Each SP-equilibrium random assignment is simply a Pseudomarket equilibrium outcome à la Hylland and Zeckhauser (1979) if $\pi = 1$. SP is indeed a combination of these two mechanisms.*

Subsequent sections explore the efficiency properties of SP. Incentive compatibility in large economies is shown for a family of mechanisms of which SP forms part in He, Miralles, Pycia and Yan (2015). What remains of this section clarifies its stability properties. We show that every SP equilibrium random assignment from an ordered partition $N_1, ..., N_\pi$ respects uniform priorities induced by such a partition.

Lemma 1 *Let Q^* be a SP-equilibrium random assignment at economy E provided an ordered partition $N_1, ..., N_\pi$. Then Q^* respects uniform priorities induced by $N_1, ..., N_\pi$ in the Kesten-Ünver ex-ante stability sense.*

Proof. Since $\sum_{x \in N_t} q_x^{i*} > 0$, it implies that $\sum_{\tau \leq r} \sum_{x \in N_\tau} q_x^{*i} < \eta^i$, for all $r < t$, and this implies $P_r^{*i} = 0$ for all $r < t$. For all agents $x \in N_1 \cup ... \cup N_{t-1}$, no object type is cheaper than i, therefore purchasing probabilities of a less-preferred object type would be suboptimal. ∎

[12] It is easy to see that this is a straightforward extension of the previous definition of ex-ante stability to uniform priorities.

[13] We assume that every Pseudomarket equilibrium satisfies the slackness condition: every object type in excess supply is sold at zero price.

[14] Without loss of generality, agents could be labeled in a way that the matrices Q^* and V are consistent (i.e. each row refers to the same agent in both matrices).

Corollary 1 *In every SP-equilibrium random assignment Q^* under the ordered partition $N_1, ..., N_\pi$, for each $t = 1, .., \pi$ and for each agent $x \in N_t$, q_x^* first-order stochastically dominates (according to x's preferences) every assignment q_y^*, if $y \in N_{t+1} \cup ... \cup N_\pi$.*

4 Welfare theorems regarding consistent weak ex-ante efficiency

The following result states that *CWEE* characterizes the set of all SP-equilibria outcomes generated by every possible ordered partition.

Theorem 1 *1) (Second Welfare Theorem for CWEE) For a finite economy E, if Q^* is CWEE, then there is an ordered partition $N_1, ..., N_\pi$ of the set N such that Q^* is a SP-equilibrium random assignment given the ordered partition $N_1, ..., N_\pi$.*

2) (First Welfare Theorem for CWEE) For each ordered partition $N_1, ..., N_\pi$ of N, every associated SP-equilibrium outcome Q^* is CWEE at E.

Proof. Part 1) It follows a recursive argument. We explain the first iteration, which is afterwards repeated with the "continuation economy" (we define it below) until all agents are removed. We start this iteration by considering a reduced economy $E^r = (N^r, \eta^r, V^r)$ that is resulting from removing all agents x who obtain a most-preferred object type: $N^M = \{x \in N : \sum_{j \in M_x} q_x^{j*} = 1\}$. We also remove their assignments from the supply vector, obtaining η^r. The remaining assignment is denoted as $Q^r = (q_x^*)_{x \in N^r}$. This is without loss of generality since any price vector would meet the competitive equilibrium condition for these agents. We also skip the simple case in which everyone obtains a most-preferred assignment.

For any agent $x \in N^r$ there exists a non-empty convex set of strictly preferred probability distributions $U_x = \{q \in \Delta^s : u_x(q) > u_x(q_x^*)\}$. Likewise, the set $U = \sum_{x \in N^r} U_x$ is well-defined and convex. Naturally, $U \subset |N^r| \cdot \Delta^s$ (since $\sum_{x \in N^r} q_x = |N^r|$). Let us define $Y = \prod_{j \in S} [0, \eta_j^r]$, the set of aggregate feasible random assignments, which is also convex. Since Q^* is *CWEE* (and so is Q^r at E^r) we have $U \cap Y = \varnothing$ (otherwise N^r would be a trading coalition). Applying the separating hyperplane theorem *to the rescaled simplex* $|N^r| \cdot \Delta^s$, there exists a price vector $P \in \mathbb{R}_+^s / \{(p, ..., p) : p \geq 0\}$ and a number $w \in \mathbb{R}$ such that $P \cdot a \geq w \geq P \cdot b$, for any $a \in U, b \in Y$. We get rid of price vectors with all equal elements since those would not divide the rescaled simplex in two parts. The object types with excess supply $(\sum_{x \in N^r} q_x^{rj} < \eta^{rj})$ would have a zero price component in any such vector P $(P^j = 0)$.

Let M be a $n \times s$ random assignment matrix (with generic element m_x^j) such that $\sum_{j \in M_x} m_x^j = 1$ for every $x \in N$. Take a random assignment Q such that $q_x \cdot v_x \geq q_x^* \cdot v_x$ $\forall x \in N$. Consider a number $\alpha \in (0, 1)$ and build the random assignment $Q^\alpha = \alpha Q + (1 - \alpha) M$. Since $q_x^\alpha \in U_x$ for every $x \in N^r$, we have $P \cdot \sum_{x \in N^r} q_x^\alpha \geq w$. Taking the limit, since $\lim\limits_{\alpha \to 1} Q^\alpha = Q$, we have $P \cdot \sum_{x \in N^r} q_x \geq w$.

The same applies to the case $Q = Q^* : P \cdot \sum_{x \in N^r} q_x^* \geq w$. But we know that $\sum_{x \in N^r} q_x^* \in Y$ because Q^* is feasible, implying $P \cdot \sum_{x \in N^r} q_x^* \leq w$. We conclude $P \cdot \sum_{x \in N^r} q_x^* = w$. For, this reason, if we take $q_x \in U_x$ for any agent $x \in N^r$, we have $P \cdot \left(q_x + \sum_{y \in N^r \setminus \{x\}} q_y^*\right) \geq w = P \cdot \left(q_x^* + \sum_{y \in N^r \setminus \{x\}} q_y^*\right)$. Consequently we have $P \cdot q_x \geq P \cdot q_x^*$, proving that P constitutes a pseudomarket quasiequilibrium for this economy E with associated random assignment Q^*.

For each agent $x \in N^r$ such that there exists a probability distribution $\bar{q}_x$ meeting $P \cdot \bar{q}_x < P \cdot q_x^*$, P is indeed a pseudomarket equilibrium price vector. This follows a standard argument. Suppose $q_x \in U_x$ and $P \cdot q_x = P \cdot q_x^*$. Take a number $\alpha \in (0,1)$ and build the random assignment $q_x^\alpha = \alpha \bar{q}_x + (1-\alpha) q_x$, which meets $P \cdot q_x^\alpha < P \cdot q_x^*$. But for α close to 0, $q_x^\alpha \in U_x$, and this would contradict the fact that P constitutes a quasi-equilibrium. Therefore we must have $P \cdot q_x > P \cdot q_x^*$, proving that P constitutes an equilibrium price vector for these agents.

We then focus on the agents for which there is no such probability distribution $\bar{q}_x$. If there is no $q_x \in U_x$ such that $P \cdot q_x = P \cdot q_x^*$, then P is indeed a quasi-equilibrium vector for this agent x. So define $N^c = \{x \in N : \exists q_x \in U_x,\ P \cdot q_x = P \cdot q_x^* = \min\limits_{j \in S} P^j\}$. If $N^c = \varnothing$ we are done since the quasiequilibrium price vector actually constitutes an equilibrium. Thus we assume $N^c \neq \varnothing$.

We claim that our partition starts by setting $N_1 = N \backslash N^c$ (the set for which P is actually an equilibrium price vector with associated random assignment $Q_1^* = [q_x^*]_{x \in N_1}$) and $N_2 \cup ... \cup N_\pi = N^c$. For this we just need to show that N_1 is not empty. If N^M is not empty, we are done. If it is, we know that there exists an "expensive" object type i such that $P^i > \min\limits_{j \in S} P^j$ (since $P \notin \{(p, ..., p) : p \geq 0\}$). If no agent x gets $q_x^{*i} > 0$, then the object type has excess supply implying $P^i = 0$, contradicting $P^i > \min\limits_{j \in S} P^j$. Therefore, some agent $x \in N$ gets $q_x^{*i} > 0$, and consequently $x \notin N^c$. Then $N \backslash N^c \neq \varnothing$ as we wanted to show.

For the next iteration, the "continuation economy" would consist of $S^c = \{j \in S : \eta^j - \sum_{x \in N_1} q_x^{*j} > 0\}$, $\eta^c = (\eta^j - \sum_{x \in N_1} q_x^{*j})_{j \in S^c}$ and N^c. We proceed as in the first iteration to find, subsequently, nonempty disjoint sets $N_2, ..., N_\pi$. For some iteration $\pi \leq n$ we have $N_1 \cup ... \cup N_\pi = N$ since N is finite, and we are done.

Part 2) It follows a recursive argument. Let a trading coalition $C \subset N$ be defined as follows: $\exists Q$ such that a) $q_x^* \cdot v_x < q_x \cdot v_x$ for all $x \in C$, b) $q_x^* = q_x$ for all $x \in N \backslash C$, and c) $\sum_{x \in C} q_x \leq \eta - \sum_{x \in N \backslash C} q_x^*$. We show that it must be the case that $C = \varnothing$.

We claim that $N_1 \cap C = \varnothing$. If not, there must be a nonempty subset $\tilde{N} \subset N_1$ and an alternative feasible random assignment Q such that $q_x^* \cdot v_x < q_x \cdot v_x$ for all $x \in \tilde{N}$ and $q_x^* = q_x$ for all $x \in N_1 \backslash \tilde{N}$. The SP-equilibrium (with price vector P_1^* associated to N_1) implies $P_1^* \cdot \sum_{x \in N_1} q_x^* < P_1^* \cdot \sum_{x \in N_1} q_x$, and therefore $\sum_{x \in N_1} q_x^{*j} < \sum_{x \in N_1} q_x^j$ for some object type j such that $P_1^{*j} > 0$. Since this price is strictly positive, there is no excess supply in the reduced economy with N_1 on the demand side and η as the supply side. We must have $\sum_{x \in N_1} q_x^{*j} = \eta^j$ and thus $\sum_{x \in N_1} q_x^j > \eta^j$. This constitutes a contradiction as Q is not feasible.

Consequently, $N_1 \cap C = \varnothing$. We focus on the "continuation economy" consisting of $S^c = \{j \in S : \eta^j - \sum_{x \in N_1} q_x^{*j} > 0\}$, $\eta^c = (\eta^j - \sum_{x \in N_1} q_x^{*j})_{j \in S^c}$ and $N \backslash N_1$. Using the same argument in each "continuation economy", we recursively see that $N_2 \cap C = \varnothing$, $N_3 \cap C = \varnothing$... Since $N = \cup_{t=1}^{\pi} N_t$, we conclude that $C = \varnothing$. ■

5 Consistent weak ex-ante efficiency and ex-ante Pareto-optimality

We ideally want to fully characterize the set of ex-ante Pareto-optimal random assignments. Since an ex-ante Pareto-optimal random assignment is CWEE, it can be generated by a SP-equilibrium for some ordered partition of the set of agents. Unfortunately, the set of SP-equilibria outcomes may not coincide with the set of ex-ante Pareto-optimal assignments. A simple example with two agents x and y and two objects i and j illustrates this fact. x is indifferent between the objects whereas y strictly prefers object i. If $N_1 = \{x\}$ and $N_2 = \{y\}$ there exists a SP-equilibrium such that x picks i and y picks the remaining object j, which is not Pareto-optimal.

Clearly, ex-ante Pareto-optimality is a finer concept of efficiency than *CWEE*. Therefore, we want to explore if the Sequential Pseudomarket can also guarantee an ex-ante Pareto-optimal random assignment.

We assume hereafter that for no agent there could be two equally valued object types.

Assumption 1: No agent is indifferent between any two object types: $v_x^i \neq v_x^j$, $\forall x \in N$, $\forall i, j \in S : i \neq j$.

This example, from Jianye Yan, illustrates that ex-ante Pareto-optimality is not guaranteed by Assumption 1 alone.

Example 1 *(By Jianye Yan). This economy has four object types* $1, ..., 4$ *with capacities* $\eta = (2, 2, 2, 1)$. *There are 3* $x-$*type agents with valuations* $v_x = (0, 1, 2, 3)$ *and 3* $y-$*type agents with valuations* $v_y = (1, 0, 2, 3)$. *All these six agents enjoy high priority (h) at all object types. There is a seventh agent, z, with valuations* $v_z = (2, 3, 1, 0)$ *and lowest priority (l) at all object types. One SP-equilibrium assignment has prices* $P_h^* = (0, 0, 3/2, 3)$. *The associated random assignment for high-priority agents is* $q_x^* = (0, 2/3, 0, 1/3)$ *and* $q_y^* = (1/3, 0, 2/3, 0)$, *yielding utility 5/3 to all six agents. For agent z, there is only one remaining unit of object type 1,* $q_z^* = (1, 0, 0, 0)$, *yielding utility 2.*

Consider this alternative feasible allocation: $q_x = (0, 1/3, 2/3, 0)$, $q_y = (2/3, 0, 0, 1/3)$, $q_z = (0, 1, 0, 0)$. *All high-priority agents still keep utility 5/3. Yet agent z is better-off, since she obtains a unit of object type 2, and payoff increases to 3.*

We immediately observe that agents with high-priority status are *indifferent* between the two allocations. Moreover, the allocation given by (q_x, q_y) is also a pseudomarket equilibrium at prices P_h^*. A deeper insight reveals a third observation: if all zero-priced object types are unified into a unique artificial one with valuation $1 = \max\{v_x^1, v_x^2\} = \max\{v_y^1, v_y^2\}$ for all high-priority agents, then agents of types x and y have linearly dependent preferences in the following sense: $\frac{v_x^4 - v_x^2}{v_x^3 - v_x^2} = \frac{v_y^4 - v_y^1}{v_y^3 - v_y^1}$. And this allowed for probability trading between x, y and z agents that left x and y types indifferent, while improving agent z's welfare. Of course, this is a rare event in the space of preference profiles. Here is a formalization of such scenarios.

For objects i, j, k and an agent x, we denote with $\rho_x(i, j, k) = \frac{v_x^j - v_x^k}{v_x^i - v_x^k}$ the marginal rate of substitution between objects i and j for agent x, after taking k as the residual alternative. This residual alternative is necessary since the agent's bundle cannot go beyond the simplex. The usual $\frac{v_x^j}{v_x^i}$ for linear utilities is not of use here.

If $s > 2$, and for the purpose of the next assumption, we can unify some object types $W \subset S$ into one object w with $\eta^w = \sum_{i \in W} \eta^i$ and $v_x^w = \max_{i \in W} v_x^i$, $x \in N$, creating a new "$W-$unification" economy $\tilde{E}$ with object types $\tilde{S} = \{w\} \cup S \backslash W$, with $|\tilde{S}| \geq 3$.

For a subset of object types $S' \subset \tilde{S}$ and $\{i, j, k\} \subset S'$, define the vector $d_x(i, j, k) \in \mathbb{R}^{|S'|}$ as: $d_x^l(i, j, k) = 0$ for $l \notin \{i, j, k\}$, $d_x^i(i, j, k) = \rho_x(i, j, k)$, $d_x^j(i, j, k) = -1$, $d_x^k(i, j, k) = 1 - \rho_x(i, j, k)$. This vector indicates the (unique) direction through which the agent's random assignment can be altered for objects i, j, k only, inside the $S'-$simplex, so that the agent remains indifferent. Every such vector is well-defined under Assumption 1.

Assumption 2: (*Regularity*) We assume that there is no $W-$unification economy $\tilde{E}$ with a subset of object types $S' \subset \tilde{S}$, and a set of $\tilde{n} = |S'|-1$ agent - object set pairs $\{x_r, \{i_r, j_r, k_r\}\}_{r=1,...,\tilde{n}}$ with $\bigcup_{r=1...\tilde{n}} \{x_r\}$ non-singleton such that $(d_{x_r}(i_r, j_r, k_r))_{r=1,...,\tilde{n}}$ are linearly dependent.

This assumption is the extension of the assumption in Bonnisseau, Florig and Jofré (2001) for linear utility economies (in their Lemma 4.1), namely that there is no cycle of marginal rates of substitution that multiplied altogether yield one. Our assumption is more involved, since it embeds a cycle of ρ_x operations, and operations inside the operations. In fact, Assumption 2 includes the assumption of Bonnisseau, Florig and Jofré (2001) as a special case. The proof is in the appendix (Lemma 2), along with an example illustrating that the converse is not true.

Remark 2 *Notice, importantly, that the set of preference profiles satisfying Assumptions 1 and 2 is **dense** in the preference profile space. Even though all vectors d have their elements adding up to zero, allowing for an elimination of one (the same) coordinate from each vector, we are remained with one $\tilde{n} \times \tilde{n}$ square matrix of corrected vectors d that is singular, if Assumption 2 is violated.*

A crucial proposition, instrumental for a subsequent theorem, arises from these assumptions.

Proposition 1 *Let a price vector P^* constitute a pseudomarket equilibrium for an economy E with associated feasible random assignment Q^*. Then, under Assumptions 1 and 2, there is no other feasible random assignment $Q \neq Q^*$ such that $q_x \cdot v_x = q_x^* \cdot v_x$ for every $x \in N$.*

Proof. In the Appendix. ■

Then the following Theorem holds:

Theorem 2 *Under Assumptions 1 and 2, if Q^* is CWEE at economy E, it is also ex-ante Pareto-optimal.*

Corollary 2 *Under Assumptions 1 and 2, every SP-equilibrium assignment Q^* for economy E is ex-ante Pareto-optimal at that economy.*

Proof. We use Theorem 1: there is an ordered partition $N_1, ..., N_\pi$ of the set N such that Q^* is a Sequential Pseudomarket equilibrium random assignment given the ordered partition $N_1, ..., N_\pi$. Consider $Q^* \in \mathcal{Q}$ not being ex-ante Pareto-optimal, thus some feasible $Q \in F_E$ ex-ante Pareto-dominates Q^*. Select $t^* = \min\{t \in \{1, ..., \pi\} : Q_{N_t} \neq Q^*_{N_t}\}$. Since Q^* is CWEE, it must be the case that $Q^*_{N_{t^*}}$ is ex-ante Pareto-optimal in the remaining economy E_{t^*} (when we only consider the agents in N_{t^*} and the supply vector is $\eta - \sum_{t<t^*} \sum_{x \in N_t} q_x$). Then $q_x \cdot v_x = q_x^* \cdot v_x$ $\forall x \in N_{t^*}$. But this is in contradiction with Proposition 1 under Assumptions 1 and 2, since $Q^*_{N_t}$ is a pseudomarket equilibrium assignment for economy E_{t^*}. ■

We could have side-stepped Assumptions 1 and 2 by imposing Pseudomarket selection rules. For instance, we could have get rid of Assumption 1 if we imposed every agent to buy the cheapest bundle among her optimal choices (Hylland

and Zeckhauser, 1979). Similarly, a well-designed equilibrium selection procedure would serve to eliminate the need for Assumption 2. Since suboptimality implies multiplicity of equilibrium allocations under the same prices for some priority group, one just needs to construct a trial and error algorithm that picks the "convenient" equilibrium allocation at each stage.

6 Conclusion

We have presented a new mechanism, Sequential Pseudomarket, that is particularly appealing in random assignment problems with object-invariant priority rights. The set of equilibrium outcomes of this mechanism is characterized by the also new concept of Consistent Weak Ex-ante Efficiency. Moreover, this concept of efficiency is generically identical to the usual ex-ante Pareto-efficiency. Altogether, an immediate application would be the random assignment problem with uniform priorities such as seniority rights. For instance, if one is interested in finding an ex-ante efficient assignment that respects seniority rights while avoiding envy among agents of the same seniority group, the simplest answer would be: run a Sequential Pseudomarket with Equal Incomes.

References

[1] Ashlagi I. and Shi P. (2015) "Optimal Allocation Without Money: an Engineering Approach," *Management Science* 62, 1078-1097.

[2] Azevedo E. and Budish E. (2015) *Strategy-proofness in the Large*, unpublished manuscript.

[3] Bonnisseau J.M., Florig M. and Jofré A. (2001) "Continuity and Uniqueness of Equilibria for Linear Exchange Economies", *Journal of Optimization Theory and Applications* 109, 237-263.

[4] Budish E., Che Y.K., Kojima F. and Milgrom P. (2012) "Designing Random Allocation Mechanisms: Theory and Applications", *American Economic Review* 103, 585-623.

[5] Chambers C.P. (2004) "Consistency in the Probabilistic Assignment Model", *Journal of Mathematical Economics* 40, 953–962.

[6] Han X. (2015) *Random Assignment Respecting Group Priorities*, unpublished manuscript.

[7] He Y., Miralles A., Pycia M. and Yan J. (2015) *A Pseudo-Market Approach to Allocation with Priorities*, unpublished manuscript.

[8] Hylland A. and Zeckhauser R. (1979) "The Efficient Allocation of Individuals to Positions", *Journal of Political Economy* 87, 293-314.

[9] Kesten O. and Ünver U. (2015) "A Theory of Random Assignment Lotteries", *Theoretical Economics* 10, 543-595.

[10] Miralles A. and Pycia M. (2014) *Prices and Efficient Assignments without Transfers*, unpublished manuscript.

[11] Roth A. and Postlewaite A. (1977) "Weak Versus Strong Domination in a Market with Indivisible Goods", *Journal of Mathematical Economics* 4, 131-137.

[12] Thomson W. (2015) *Consistent Allocation Rules*, unpublished manuscript.

[13] Thomson W. and Zhou L. (1993) “Consistent Allocation Rules in Atomless Economies”, *Econometrica* 61, 575-587.

7 Appendix

7.1 Proof of Proposition 1

Let a price vector P^* constitute a pseudomarket equilibrium for an economy E with associated feasible random assignment Q^*. Then, under Assumptions 1 and 2, there is no other feasible random assignment $Q \neq Q^*$ such that $q_x \cdot v_x = q_x^* \cdot v_x$ for every $x \in N$.

Proof. Take a feasible random assignment $Q \neq Q^*$ such that $q_x \cdot v_x = q_x^* \cdot v_x$ $\forall x \in N$.

Since by Assumption 1 no agent x is indifferent between any two objects, we cannot have $P^* \cdot q_x < P^* \cdot q_x^*$ for any $x \in N$, provided $q_x \cdot v_x = q_x^* \cdot v_x$. (Else q_x^* would not be an optimal choice with prices P^*: being M_x x's certain assignment to her unique most preferred object type, and for $\alpha > 0$ small enough, $\alpha M_x + (1-\alpha)q_x$ would be a better and affordable choice). Therefore $P^* \cdot q_x \geq P^* \cdot q_x^*$ for all $x \in N$. On the other hand, we cannot have $P^* \cdot q_x > P^* \cdot q_x^*$ for any $x \in N$. Otherwise we would have $\sum_{x\in N} q_x > \sum_{x\in N} q_x^*$ for some object type i such that $P^{i*} > 0$. Since this price is positive, it must be the case that $\sum_{x\in N} q_x^* = \eta^i$. Hence Q is not feasible, a contradiction. We conclude that $P^* \cdot q_x = P^* \cdot q_x^*$ for all $x \in N$. That is, Q is an equilibrium assignment associated to P^*.

Let $W = \{i \in S : P^{i*} = 0\}$. Unify all object types with zero price as the same object type w. Its supply is $\eta^w = \sum_{i\in W} \eta^i$. Each agent x's valuation for this object is $\tilde{v}_x^w = \max_{i\in W} v_x^i$, $x \in N$. Valuations for the remaining objects are unaltered: $\tilde{v}_x^j = v_x^j$ whenever $j \notin W$. Consider such a $W-$unified economy with object types $\tilde{S} = \{w\} \cup S \backslash W$. Obviously, there is a competitive equilibrium in this economy with prices equal to $\tilde{P}^{i*} = P^{i*}$, $i \neq w$, and $\tilde{P}^{w*} = 0$. Equilibrium assignments are $\tilde{q}_x^{w*} = \sum_{i\in W} q_x^{i*}$ and $\tilde{q}_x^{i*} = q_x^{i*}$, $i \neq w$, for assignment $\tilde{Q}^*$ analogous to Q^*. An identical transformation yields $\tilde{Q}$ from Q. Notice that $Q \neq Q^*$ implies $\tilde{Q} \neq \tilde{Q}^*$, provided Assumption 1. No differences in the assignments Q and Q^* can only arise from differences in the assignments of the free goods, since this latter fact is only possible when some agent is indifferent between two free goods.

From now on we assume that $|\tilde{S}| > 2$. If $|\tilde{S}| = 1$ this would directly negate $\tilde{Q} \neq \tilde{Q}^*$. If $|\tilde{S}| = 2$ then for each agent the optimal choice is unique: either picking the free good for sure, or combining the non-free good with the free good if necessary. No indifference between these two options is possible since indifference between them arises only if the agent is indifferent between the free object and the non-free object. Once again, this would contradict $\tilde{Q} \neq \tilde{Q}^*$.

Denote with A the (nonempty) set of agents x such that $\tilde{q}_x \neq \tilde{q}_x^*$. For each $x \in A$, let $S_x = \{i \in \tilde{S} : \tilde{q}_x^i + \tilde{q}_x^{i*} > 0\}$, the set of objects with positive demand at either or both allocations. The binding budget constraint guarantees that $|S_x| \geq 3$ for each $x \in A$. (Either $\tilde{q}_x^i$ or $\tilde{q}_x^{i*}$ or both contain at least two object types with positive purchased probabilities. Assumption 1 ensures that only one optimally chosen bundle may consist of a sure allocation of one object type). Since $\tilde{q}_x \cdot \tilde{v}_x = \tilde{q}_x^* \cdot \tilde{v}_x$ and $\tilde{P}^* \cdot \tilde{q}_x = \tilde{P}^* \cdot \tilde{q}_x^* \; \forall x \in A$, for each $x \in A$ there is $\alpha_x, \beta_x \geq 0$ such that for any $i \in S_x$ we have $\tilde{v}_x^i = \alpha_x + \beta_x \tilde{P}_{t^*}^{i*}$. Particularly, this implies that for any triple $\{i, j, k\} \subset S_x$ we have $\rho_x(i,j,k) \equiv \frac{\tilde{v}_x^j - \tilde{v}_x^k}{\tilde{v}_x^i - \tilde{v}_x^k} = \frac{\tilde{P}^{j*} - \tilde{P}^{k*}}{\tilde{P}^{i*} - \tilde{P}^{k*}}$. Under Assumption 1 (no pairwise indifference), this is always well-defined, since $\tilde{P}^{i*} = \tilde{P}^{k*}$ is in contradiction with both i and k being purchased with positive probability. Let Σ_x denote the collection of all three-element sets in $S_x : \Sigma_x = \{\sigma = \{i,j,k\} \subset S_x\}$.

For each $\sigma \in \Sigma_x$ let δ_σ be the only direction in the $\tilde{S}$–simplex in which one can modify quantities of only objects in $\sigma = \{i,j,k\}$ along the budget frontier (i.e. $\delta_\sigma \cdot 1_{|\tilde{S}|} = 0$ and $\delta_\sigma \cdot \tilde{P}^* = 0$): $\delta_\sigma^i = \frac{\tilde{P}^{j*} - \tilde{P}^{k*}}{\tilde{P}^{i*} - \tilde{P}^{k*}}$, $\delta_\sigma^j = -1$, $\delta_\sigma^k = 1 - \frac{\tilde{P}^{j*} - \tilde{P}^{k*}}{\tilde{P}^{i*} - \tilde{P}^{k*}}$, $\delta_\sigma^l = 0$ for all $l \notin \sigma$ (δ_σ is well-defined under Assumption 1: recall that $\tilde{P}^{i*} = \tilde{P}^{k*} \Longrightarrow \sigma \notin \Sigma_x$ for any $x \in A$) Since $\sum_{x \in A}(\tilde{q}_x - \tilde{q}_x^*) = 0$ after the preceding W–unification, the components of that sum can be ordered in a path $(\tilde{q}_x - \tilde{q}_x^*)_{x \in A}$ that starts and ends at the origin. We must then have at least one finite set of pairs agent - object sets $(\{x_r, \sigma_r\})_{\sigma_r \in \Sigma_{x_r}, r=1\ldots T}$ that induce a collection of linearly dependent vectors $\Delta = \{\delta_{\sigma_r}\}_{\sigma_r \in \Sigma_{x_r}, r=1\ldots T}$. Should all elements in $(\delta_\sigma)_{\sigma \in \Sigma_x, x \in A}$ be linearly independent, there would be no path $(\tilde{q}_x - \tilde{q}_x^*)_{x \in A}$ from the origin back to the origin with one-shot moves along different, linearly independent directions.

We can find a "multi-agent" Δ in the sense that $\bigcup_{r=1\ldots T} \{x_r\}$ is not a singleton. We show this by contradiction. Let $X \subset A$ be the set of agents such that for each $x \in X$, the collection of elements in $(\delta_\sigma)_{\sigma \in \Sigma_x}$ is linearly dependent, but none of its elements is linearly independent from $(\delta_\sigma)_{\sigma \in \Sigma_y, y \in A \setminus \{x\}}$. By way of contradiction, $X \neq \varnothing$, and the collection $(\delta_\sigma)_{\sigma \in \Sigma_x, y \in A \setminus X}$ would contain linearly independent vectors, among themselves and also with respect to $(\delta_\sigma)_{\sigma \in \Sigma_y, y \in A \setminus X}$. But then, since the path $(\tilde{q}_x - \tilde{q}_x^*)_{x \in A}$ starts and ends at the origin, we must conclude that $A = X$. But, again, since for each $x \in X$, the collection of elements in $(\delta_\sigma)_{\sigma \in \Sigma_x}$ is linearly independent from $(\delta_\sigma)_{\sigma \in \Sigma_y, y \in X \setminus \{x\}}$, each agent in X will have her own isolated path starting and ending at the origin, that is, $\tilde{q}_x - \tilde{q}_x^* = 0$ for all $x \in X$. This contradicts the definition of A.

In the set D of all such "multi-agent" Δ's, we focus on some $\Delta^* \in \arg\min_{\Delta \in D} |\Delta|$ with the minimum number of vectors, a number we denote with $\tilde{n}$. Let $S' = \bigcup_{\{\delta_{\sigma_r}\} \in \Delta^*} \sigma_r$. Notice that $m \notin S'$ implies that the corresponding coordinate for m is zero for every vector in Δ^*. Then, provided the two constraints $\delta_\sigma \cdot 1_{|\tilde{S}|} = 0$ and $\delta_\sigma \cdot \tilde{P}^* = 0$, there are at most $|S'| - 2$ independent vectors in Δ^*. Actually, there are exactly $|S'| - 2$ independent vectors, since $\tilde{n}$ is minimal. This implies $\tilde{n} = |S'| - 2 + 1$, or $|S'| = \tilde{n} + 1$.

Now notice that for a collection of vectors $\{d_\sigma \in \mathbb{R}^{|S'|}, \sigma = (i,j,k)\}_{\delta_\sigma \in \Delta}$ meeting: $d_\sigma^l = 0$ for $l \notin \sigma$, $d_\sigma^i = \frac{\tilde{P}^{j*} - \tilde{P}^{k*}}{\tilde{P}^{i*} - \tilde{P}^{k*}}$, $d_\sigma^j = -1$, $d_\sigma^k = 1 - \frac{\tilde{P}^{j*} - \tilde{P}^{k*}}{\tilde{P}^{i*} - \tilde{P}^{k*}}$, this collection is linearly dependent (in comparison to Δ^*, we have only erased coordinates $m \notin S'$.) Finally, notice that $\frac{\tilde{P}^{j*} - \tilde{P}^{k*}}{\tilde{P}^{i*} - \tilde{P}^{k*}} = \rho_x(i,j,k)$ for every $x \in A$ such that $\{i,j,k\} \in \Sigma_x$. This concludes the proof, since we are contradicting Assumption 2. ■

7.2 Illustrations of Assumption 2

Assumption 2: (*Regularity*) We assume that there is no $W-$unification economy $\tilde{E}$ with a subset of object types $S' \subset \tilde{S}$, and a set of $\tilde{n} = |S'| - 1$ agent - object set pairs $\{x_r, \{i_r, j_r, k_r\}\}_{r=1,...,\tilde{n}}$ such that $(d_{x_r}(i_r, j_r, k_r))_{r=1,...,\tilde{n}}$ are linearly dependent.

We claim that this assumption can embed the assumption that bans multiplicity of equilibria in linear utility economies. Indeed, there is no difference between our model and a linear utility model when there is only one object type that is affordable for every agent, which we call w. This object has zero price in equilibrium.

Lemma 2 *For an economy E, let a Pseudomarket equilibrium price vector P^* have an associated random assignment Q^* such that $P^{i*} > P^* \cdot q_x^* > P^{w*} = 0$, $\forall i \in S \backslash \{w\}$, $\forall x \in N$. Then there is no other feasible assignment $Q \neq Q^*$ such that $q_x \cdot v_x = q_x^* \cdot v_x$ for every $x \in N$ if there is no cycle of agents and object types $(\{x_r, i_r\})_{r=1,...,\tilde{n}}$ (with not all agents nor all objects identical) such that*

$$\rho_{x_1}(i_1, i_2, w) \cdot \rho_{x_2}(i_2, i_3, w) \cdot ... \cdot \rho_{x_{\tilde{n}-1}}(i_{\tilde{n}-1}, i_{\tilde{n}}, w) \cdot \rho_{x_{\tilde{n}}}(i_{\tilde{n}}, i_1, w) = 1$$

Proof. We ignore the agents whose favorite object type is w. They obtain sure assignment of this object in both allocations. The rest of agents have to choose among the different combinations of some object $i \neq w$ and w. Hence is it without loss of generality in this setup that we focus on $d_x(i, j, k)$ such that $k = w$. Under Assumption 2, there is no $\tilde{n} \times (\tilde{n} + 1)$ matrix (where at least one agent is different)

$$\begin{bmatrix} 1 - \rho_{x_1}(i_1, i_2, w) & \rho_{x_1}(i_1, i_2, w) & -1 & 0 & \cdots & 0 \\ 1 - \rho_{x_2}(i_2, i_3, w) & 0 & \rho_{x_2}(i_2, i_3, w) & -1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 1 - \rho_{x_{\tilde{n}-1}}(i_{\tilde{n}-1}, i_{\tilde{n}}, w) & 0 & 0 & \ddots & \rho_{x_{\tilde{n}-1}}(i_{\tilde{n}-1}, i_{\tilde{n}}, w) & -1 \\ 1 - \rho_{x_{\tilde{n}}}(i_{\tilde{n}}, i_1, w) & -1 & 0 & \cdots & 0 & \rho_{x_{\tilde{n}}}(i_{\tilde{n}}, i_1, w) \end{bmatrix}$$

with rank lower than $\tilde{n}$. It means that the determinant of this matrix after we eliminate the first column is zero. And this determinant is precisely

$$\rho_{x_1}(i_1, i_2, w) \cdot \rho_{x_2}(i_2, i_3, w) \cdot ... \cdot \rho_{x_{\tilde{n}-1}}(i_{\tilde{n}-1}, i_{\tilde{n}}, w) \cdot \rho_{x_{\tilde{n}}}(i_{\tilde{n}}, i_1, w) - 1$$

proving the desired result. ■

Notice that, if we normalize valuations by subtracting v_x^w from all valuations of agent x, and we do it for all $x \in N$, we obtain the same condition as in Lemma 4.1 in Bonnisseau, Florig and Jofré (2001).

Assumption 2 in practice: a more complex example

We complete the appendix with an elaborate example that illustrates how Assumption 2 generally translates into a more complex relation among agents' preferences. Consider the following matrix:

$$\begin{bmatrix} 1 - \rho_1(b, c, a) & \rho_1(b, c, a) & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & \rho_2(d, a, e) & 1 - \rho_2(d, a, e) & 0 \\ 0 & -1 & 0 & \rho_3(d, b, f) & 0 & 1 - \rho_3(d, b, f) \\ 0 & 0 & -1 & 0 & 1 - \rho_4(f, c, e) & \rho_4(f, c, e) \\ 0 & 0 & 0 & \rho_5(d, e, f) & -1 & 1 - \rho_5(d, e, f) \end{bmatrix}$$

After erasing the last column, its determinant is

$$\rho_1(b,c,a)\rho_2(d,a,e) - \rho_2(d,a,e) - \rho_1(b,c,a)\rho_3(d,b,f) + \rho_1(b,c,a)\rho_5(d,e,f)$$
$$+\rho_2(d,a,e)\rho_5(d,e,f) - \rho_4(f,c,e)\rho_5(d,e,f) - \rho_1(b,c,a)\rho_2(d,a,e)\rho_5(d,e,f)$$

Since colinearity implies that this expression is zero, we can solve for $\rho_1(b,c,a)$ as

$$\begin{aligned}\rho_1(b,c,a) &= \frac{-\rho_4(f,c,e)\frac{\rho_5(d,e,f)}{1-\rho_5(d,e,f)} - \rho_2(d,a,e)}{\frac{\rho_3(d,b,f)-\rho_5(d,e,f)}{1-\rho_5(d,e,f)} - \rho_2(d,a,e)} \\ &= \frac{\rho_4(f,c,e)\rho_5(d,f,e) - \rho_2(d,a,e)}{\frac{\rho_3(d,b,f)-\rho_5(d,e,f)}{1-\rho_5(d,e,f)} - \rho_2(d,a,e)}\end{aligned}$$

For the second equality, notice that $1-\rho_x(i,j,k) = \rho_x(k,j,i)$, and $\rho_x(i,j,k)/\rho_x(k,j,i) = -\rho_x(i,k,j)$. This example illustrates that Assumption 2 can be expressed as a condition on a chain of multiplications of marginal rates of substitutions only in very limited cases. In general, a violation of Assumption 2 implies that one marginal rate of substitution (with a third alternative included) can be expressed as a chain of operators of the same type. In fact, for our example, one could create an imaginary agent y with preferences such that $\rho_y(b,c,a) = \rho_1(b,c,a)$, $\rho_y(d,a,e) = \rho_2(d,a,e)$, $\rho_y(d,b,f) = \rho_3(d,b,f)$, $\rho_y(f,c,e) = \rho_4(f,c,e)$, and $\rho_y(d,e,f) = \rho_5(d,e,f)$. One could rapidly check that

$$\begin{aligned}\rho_y(b,c,a) &= \frac{\rho_y(f,c,e)\rho_y(d,f,e) - \rho_y(d,a,e)}{\frac{\rho_y(d,b,f)-\rho_y(d,e,f)}{1-\rho_y(d,e,f)} - \rho_y(d,a,e)} \\ &= \rho_y[\rho_y[\rho_y(b,b,f),\rho_y(d,b,f),\rho_y(d,e,f)],\rho_y[\rho_y(f,d,e),\rho_y(f,c,e),\rho_y(f,e,e)],\rho_y(d,a,e)]\end{aligned}$$

For the second equality we use the tricks $\rho_y(i,i,j) = 1$, $\rho_y(i,j,j) = 0$ and $\rho_y(i,j,k) = 1/\rho_y(j,i,k)$. Notice that the ρ_y operator appears inside another ρ_y operator, and so on.