Decision making under risk: Lotteries

1. Let $\succsim$ satisfy completeness, transitivity, and the Independence axiom on a set Π. Prove that for any two alternatives $x, y \in \Pi$ with $x \succsim y$ and for any $1 > \alpha > \beta > 0$:

$$\alpha x + (1-\alpha)y \succsim \beta x + (1-\beta)y.$$

> **Solution:** By $x \succsim y$ and the independence axiom
>
> $$\alpha x + (1-\alpha)y \succsim \alpha y + (1-\alpha)y \sim y.$$
>
> Moreover, applying it again, we use
>
> $$\alpha x + (1-\alpha)y \sim \frac{\beta}{\alpha}(\alpha x + (1-\alpha)y) + (1-\frac{\beta}{\alpha})(\alpha x + (1-\alpha)y)$$
>
> and
>
> $$\frac{\beta}{\alpha}(\alpha x+(1-\alpha)y)+(1-\frac{\beta}{\alpha})(\alpha x+(1-\alpha)y) \succsim \frac{\beta}{\alpha}(\alpha x+(1-\alpha)y)+(1-\frac{\beta}{\alpha})y \sim \beta x+(1-\beta)y.$$

2. Consider an agent whose preferences satisfy the Independence Axiom.

   (a) Consider four lotteries $p, q, r, s \in \Delta(X)$ over prizes in $X = \{x, y, z\}$ with $p = (p(x), p(y), p(z))$, etc.

   $p = (0.2, 0.3, 0.5)$,
   $q = (0.25, 0.35, 0.4)$,
   $r = (0.8, 0, 0.2)$,
   $s = (0.9, 0.1, 0)$.

   When you learn $p \succsim q$, what can you infer about the ranking of r relative to s?

   > **Solution:** Look for each $x \in X$ for the greatest common component $\min\{p(x), q(x)\}$ to obtain $(0.2, 0.3, 0.4)$, normalize to lottery $k = \frac{1}{0.9}(0.2, 0.3, 0.4)$ such that $p = 0.9k + 0.1(0, 0, 1)$ and $q = 0.9k + 0.1(0.5, 0.5, 0)$. By the independence axiom $(0, 0, 1) \succ (0.5, 0.5, 0)$. Now do the same thing for r and s: $\min\{r(x), s(x)\}$ yields $(0.8, 0, 0)$. Thus $r = 0.8(1, 0, 0) + 0.2(0, 0, 1)$ and $s = 0.8(1, 0, 0) + 0.2(0.5, 0.5, 0)$. Hence, by the above and the independence axiom $r \succ s$.

   (b) For the same lotteries, suppose that sure prizes can be ranked such that $\delta_z \succsim \delta_y \succsim \delta_x$. Show that $p \succsim_{FSD} q$.

Solution: FSD requires that $F_q(b) - F_p(b) = \sum_{a \preceq b}(q(a) - p(a)) \geq 0$ for all $b \in X$. In our case $0.25 = q(x) = F_q(x) > F_p(x) = p(x) = 0.2$ and $0.5 = p(z) = (1 - F_p(z)) > (1 - F_q(z)) = q(z) = 0.4$. Which proves FSD.

(c) Verify that the Independence axiom implies a preference for FSD-dominant lotteries by showing that the axiom indeed implies $p \succsim q$.

Solution: Note that $\frac{p(x)}{q(x)}\delta_x + \frac{q(x)-p(x)}{q(x)}\delta_y \succsim \delta_x$. Moreover $\frac{p(z)-q(z)}{p(z)}\delta_y + \frac{q(z)}{p(z)}\delta_z \succsim \delta_y$. Finally, we have

$$p \sim q(x)[\frac{p(x)}{q(x)}\delta_x + \frac{q(x)-p(x)}{q(x)}\delta_y] + q(y)[\frac{p(z)-q(z)}{p(z)}\delta_y + \frac{q(z)}{p(z)}\delta_z] + q(z)[\delta_z].$$

Applying the IA yields $p \succsim q$

3. Determine whether the following utility criteria satisfy the axioms of expected utility:

   1. Preference for "greater certainty": $v(p) = \max_{x \in X} p(x)$.
   2. The agent considers a subset $G \subseteq X$ "good" outcomes. He ranks lotteries by the total probability of a good outcome: $v(p) = \sum_{x \in G} p(x)$.
   3. Judge by worst case: $v(p) = \min_{x \in X}\{u(x) | p(x) > 0\}$.
   4. Judge by most likely prize: $v(p) = \arg\max_{x \in X} p(x)$.

Solution:

1. Fails independence. E.g. With lotteries $p = (0.7, 0.3), q = (0.3, 0.7)$, we have $p \succsim q$, but $\alpha q + (1-\alpha)q \succ \alpha p + (1-\alpha)q$ for all $\alpha \in (0,1)$.

2. Is fine, like an expected utility maximizer who is indifferent between all outcomes in G and indifferent between all outcomes outside of G.

3. Fails continuity. Take the case $X = x, y, z$ with $\delta_x \succ \delta_y \succ \delta_z$. Lottery $p = (0,0,1), q = (0,1,0)$ and $r = (1,0,0)$. There is no $\alpha \in [0,1]$ such that $\alpha p + (1-\alpha)r \sim q$. Also fails independence.

4. Fails independence. Take the lotteries from above. While $q \succ p$, we get $\alpha q + (1-\alpha)r \succsim \alpha p + (1-\alpha)r$ for all $\alpha > 0.5$.

4. Suppose two EU maximizers with von Neumann-Morgenstern utility functions u_1 and u_2 with $u_2 = \phi \circ u_1$.

(a) Show that $\phi' > 0, \phi'' < 0$ implies that at all wealth levels w the degree of absolute risk aversion of 2 is greater than that of 1.

Solution:

$$A_2(x) = -\frac{\phi''(u_1(x))u_1'(x) + \phi'(u_1(x))u_1''(x)}{\phi'(u_1(x))u_1'(x)}$$

$$A_2(x) = -\frac{\phi''(u_1(x))}{\phi'(u_1(x))} + A_1(x)$$

Whenever ϕ' is positive, $A_2(x) \geq A_1(x)$ IFF ϕ'' is nonpositive.

(b) Show that $\phi' > 0, \phi'' < 0$ implies that 2 is more risk-averse in the sense of Arrow and Pratt.

Solution: Suppose $E(\widetilde{\epsilon}) = 0$. We want to show that $Eu_1(w+\widetilde{\epsilon}) \leq u_1(w)$ implies $Eu_2(w+\widetilde{\epsilon}) \leq u_2(w)$. $Eu_2(w+\widetilde{\epsilon}) = E\phi(u_1(w+\widetilde{\epsilon}) \leq \phi(Eu_1(w+\widetilde{\epsilon})) \leq \phi(u_1(w)) = u_2(w)$. The inequalities just use *Jensen's inequality* to get $Ef(\widetilde{x}) \leq fE(\widetilde{x})$.

Recommended Exercise. (No need to hand in)

5. Consider an EU maximizer with vNM function $u(x) = 2\sqrt{x}$ and a fair coin flip. If heads show up she gets 71, if tails show up she gets 15.

(a) Determine the risk premium associated to this gamble at wealth level 10.

Solution:

$$Eu(w+\widetilde{x}) = [\sqrt{10+71} + \sqrt{10+15}] = 14$$

$$u^{-1}(14) = (\frac{14}{2})^2 = 49 = 10 + 39$$

$$E\widetilde{x} = 43$$

Hence the risk premium is $43 - 39 = 4$.

(b) Calculate the degrees of absolute and relative risk aversion at wealth levels w. Would the risk premium change if wealth decreased to 1?

Solution: This is a CRRA function with constant relative risk aversion $R(w) = 0.5$ and absolute risk aversion $A(w) = 0.5/w$. The risk premium stays the same for all wealth levels IFF utility is CARA, here we have DARA ($A'(w) < 0$), so NO, the risk premium increases.