Sequential Pseudomarkets and the Weak Core in Random Assignments

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Abstract

We study random assignment economies with expected-utility agents, each of them eventually obtaining a single object. The assignment should respect object-invariant priorities such as seniority rights in student residence assignment, grandfather rights in landing slot assignment... We introduce the new Sequential Pseudomarket (SP) mechanism, where the set of agents is partitioned into ordered priority groups that are called in turns to participate in a pseudomarket for the remaining objects. We show that the set of all SP-equilibrium random assignments generated by every possible ordered partition coincides with the ex-ante weak core. Moreover, if we fix priority groups and individual budget limits, the resulting SP-equilibrium assignments are generically ex-ante Pareto-optimal.

Keywords: Random assignment, ex-ante efficiency, weak core, sequential pseudomarket
JEL codes: D47, D50, D60

1 Introduction

In a random assignment, each agent is provided with a probability distribution over the set of object types. Agents have preferences over their assigned distributions according to the expected utility form. No monetary transfers are allowed. Hylland and Zeckhauser's (1979) seminal paper suggests that a pseudomarket can be constructed in which each agent is endowed by some artificial income

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with which she can buy assignment probabilities. Each object type is given a nonnegative price and each agent buys a proper probability distribution (probabilities add up to 1) over them. Given the endowment vector, there is at least one equilibrium price vector yielding a feasible random assignment as an outcome. Moreover, this random assignment is ex-ante Pareto-efficient, in a sort of First Theorem of Welfare Economics for random assignment economies.

However, preexisting priority rights are found in many assignment problems. For example, in school choice, a child whose parents apply for the last slot at a public school cannot typically occupy it if the parents of another child with a sibling already attending the school want that slot (the so-called sibling priority). There are many priority criteria in many different assignment problems: proximity to the school, low income, being organ donor in "kidney exchange"... This paper especially takes into account priority structures that are object-invariant, that is, independent of the object for which agents are competing. Examples of this kind of priorities are seniority rights in the assignment of students to college residences, or "grandfather rights" in the assignment of landing slots and gates in airports. *The motivation question of this note is whether there is a mechanism that respects object-invariant priorities while it attains good ex-ante efficiency properties.*

We first introduce a new mechanism, the *Sequential Pseudomarket* (SP). In SP, ordered groups of agents (top-priority agents, second-priority agents...) are called in turns that attend the pseudomarket for the remaining objects. A SP-equilibrium is a sequence of pseudomarket equilibria turn by turn. It is easy to see that SP ecompasses a family of mechanisms in whose oposite extremes we find both serial dictatorship and pseudomarkets.

We define the *ex-ante weak core* as the set that contains all feasible random assignments such that, for each one, there is no coalition of agents and redistribution of probabilities across them in which all agents in the coalition are ex-ante strictly better-off. The ex-ante weak core is a subset of the set of all randomizations over strong-domination stable allocations, using the terminology of Roth and Postlewaite (1977). The ex-ante weak core is hence the extension of Roth and Postlewaite's notion of stability to random assignments.[1] In an important characterization result, we show that *the ex-ante weak core is equal to the set of all random assignments generated by SP-equilibria* for every partition of the set of agents into ordered groups (Theorem 1).

Understandably, the previous characterization result assumes noth-

[1] A final allocation is strong-domination stable if it is in the weak core of a market were the final allocation is taken as the endowment. We did not use the name of ex-ante stability since it has already ben used in the literature for a different purpose (Kesten and Ünver, forthcoming).

ing a priori about the agents' budget limits. There are reasons for which we should impose conditions on them. For instance, one could argue that no ex-ante envy might arise among agents of the same priority type, hence all of them should have the same budget limit. Or other considerations could affect the way budget limits are chosen. A similar issue arises with priorities. Pre-fixed, exogenous criteria typically lead the partition of the set of agents into priority groups.

So we provide the following additional result (Theorem 2): *for fixed budgets and priority groups, we can generically state that every arising SP-equilibrium random assignment is ex-ante Pareto-optimal.* Only two zero-measure events preclude this statement from being certain. First, that the economy is not regular, in the sense of yielding a continuum of SP-equilibria. Second, that some agent is indifferent between two object types. SP with fixed turns and budgets appears as a very comprehensible mechanism that obtains good efficiency properties while respecting preexisting priorities. This result is somewhat striking since differently ranked agents face different relative prices, a fact that could have caused inefficiency of the random assignment.

This note takes a general equilibrium theory approach to the stochastic assignment of indivisible goods. Apart from Hylland and Zeckhauser (1979), other reference in this strand of literature is Mas-Colell's (1992). He provides a general result for every kind of economy with possibly satiated preferences, including assignment economies. For any ex-ante Pareto-optimal allocation, he shows the existence of a Walrasian equilibrium with slack supporting it. It is still to be proven that no slack is necessary. Our paper provides no need for slack when Pareto-optimality is relaxed. In exchange, we need the agents to be weakly ordered into turns to enter the market.

More than thirty years after the seminal paper by Hylland and Zeckhauser, pseudomarkets are attracting increasing interest both in finite and continuum economies.[2] Examples of recent papers are Budish and Azevedo (2012) on strategy-proofness in the large that applies to pseudomarkets, or Budish, Che, Kojima and Milgrom (2012) on pseudomarket mechanisms for multidimensional assignment. We contribute to this literature by providing a proper and simple combination between pseudomarket and serial dictatorship that performs satisfactorily in assignment problems with object-invariant priority structures.

2 The model

In this economy there is a finite set of agents $N = \{1, ..., n\}$. The

[2] See Thomson and Zhou (1993) for a result on efficient and fair allocations in continuum economies.

notation $x, y...$ is used for a generic element of N. There is a set of object types $S = \{1, ..., s\}$. The notation $i, j...$ serves to indicate a generic element of S. For each object type j there is a number of identical copies $\eta^j \in \mathbb{N}$. $\eta = (\eta^1, ..., \eta^s)$ is the supply in this economy. We have enough supply in the sense that $\sum_{j \in S} \eta^j \geq n$.[3]

A random assignment is a $n \times s$ matrix Q whose generic element $q_x^j \geq 0$ is the probability that agent x obtains a copy of object type j. This matrix is stochastic: $\sum_{j \in S} q_x^j = 1$ for any $x \in N$. Agent x's random assignment is the probability distribution $q_x = (q_x^1, ..., q_x^s) \in \Delta^s$ (Δ^s is the $s-1$ dimensional simplex). A random assignment is a pure assignment if each of its elements is either 1 or 0. A random assignment is feasible if $Q' \cdot 1_n \leq \eta$ (where 1_n is a vector of n ones and $'$ denotes the transpose of a matrix). A feasible random assignment can be expressed as a lottery over feasible pure assignments.

Let $V \in \mathbb{R}_+^{ns}$ denote a $n \times s$ matrix of nonnegative von Neumann-Morgenstein valuations, whose generic element v_x^j indicates agent x's valuation for object type j. A generic agent x's valuation vector is $v_x = (v_x^1, ..., v_x^s)$. She values her random assignment q_x as the vectorial product $u_x(q_x) = v_x \cdot q_x$. Each agent x has a set of most-preferred object types $M_x = \underset{j \in S}{\arg\max} v_x^j$. An economy is a triple $E = (N, \eta, V)$.

Let $\mathcal{F}_E$ denote the set of feasible random assignments in an economy E. A feasible random assignment Q^{PO} is *ex-ante Pareto-optimal* at an economy E if for any random assignment Q, $diag(VQ') > diag(VQ'_{PO}) \Longrightarrow Q \notin \mathcal{F}_E$. ($diag$ denotes the diagonal of a matrix, and $>$ indicates that the inequality is strict for at least one element).

Considering a feasible random assignment Q^{CO}, let a blocking coalition $C \subset N$ be defined as follows: $\exists Q$ such that a) $q_x^{CO} \cdot v_x < q_x \cdot v_x$ for all $x \in C$, b) $q_x^{CO} = q_x$ for all $x \in N \backslash C$, and c) $\sum_{x \in C} q_x \leq \eta - \sum_{x \in N \backslash C} q_x^{CO}$. A feasible random assignment Q^{CO} belongs to the *(ex-ante) weak core* of an economy E if its unique blocking coalition is $C = \varnothing$.

A price vector is notated as $P \in \mathbb{R}_+^s$. A price vector P^* constitutes a pseudomarket *quasiequilibrium* for an economy E with associated feasible random assignment Q^* if for any random assignment Q and any agent x we have $u_x(q_x) > u_x(q_x^*) \Longrightarrow P^* \cdot q_x \geq P^* \cdot q_x^*$. A price vector P^* constitutes a pseudomarket *equilibrium* for an economy E with associated feasible random assignment Q^* if for any random assignment Q and any agent x we have $u_x(q_x) > u_x(q_x^*) \Longrightarrow P^* \cdot q_x > P^* \cdot q_x^*$. A price vector P^* constitutes a pseudomarket *equilibrium with budget limits* $(B_x)_{x \in N}$ for an economy E with associated feasible random assignment Q^* if for any random assignment Q and any agent x we have both $P^* \cdot q_x^* \leq B_x$ and $u_x(q_x) > u_x(q_x^*) \Longrightarrow P^* \cdot q_x > B_x$.

[3]Notice that the weak inequality allows for an easy inclusion of an outside option for every agent.

3 Sequential pseudomarkets and the weak core

Let the set N be partitioned into disjoint ordered sets $N_1, ..., N_\pi$ with $\pi \leq n$. Start with a reduced economy E_1 with N_1 on the demand side and $\eta_1 = \eta$ as the supply side. Calculate a pseudomarket equilibrium allocation Q_1^* for this reduced economy. For $t = 2, .., s$, calculate the remaining supply $\eta_t = \eta_{t-1} - Q_{t-1}^{*\prime} \cdot 1_{|N_{t-1}|}$ and use N_t on the demand side to calculate a new pseudomarket equilibrium allocation $Q_t^*$for the reduced economy $E_t = (N_t, \eta_t, V_t)$. (V_t is a selection of V that contains the preferences for agents in N_t) The vertical composite matrix $Q^* = [Q_1^*, ..., Q_\pi^*]$ constitutes a Sequential Pseudomarket (SP) equilibrium random assignment given the ordered partition $N_1, ..., N_\pi$.[4]

When $\pi = n$ we have a Serial Dictatorship, whereas on the other extreme we have a Pseudomarket equilibrium outcome if $\pi = 1$. SP is indeed a combination of these two mechanisms.

The following result states that the set of all SP-equilibria outcomes generated by every possible ordered partition coincides with the (ex-ante) weak core.

Theorem 1 *1) For a finite economy E, if Q^* belongs to the weak core, then there is an ordered partition $N_1, ..., N_\pi$ of the set N such that Q^* is a Sequential Pseudomarket equilibrium random assignment given the ordered partition $N_1, ..., N_\pi$.*

2) Moreover, for each ordered partition $N_1, ..., N_\pi$ of N, every associated Sequential Pseudomarket equilibrium outcome Q^* belongs to the weak core of E.

Proof. Part 1) It follows a recursive argument. We explain the first iteration, which is afterwards repeated with the "continuation economy" (we define it below) until all agents are removed. We start this iteration by considering a reduced economy $E^r = (N^r, \eta^r, V^r)$ that is resulting from removing all agents x who obtain a most-preferred object type: $N^M = \{x \in N : \sum_{j \in M_x} q_x^{j*} = 1\}$. We also remove their assignments from the supply vector, obtaining η^r. The remaining assignment is denoted as $Q^r = (q_x^*)_{x \in N^r}$. This is without loss of generality since any price vector would meet the competitive equilibrium condition for these agents. We also skip the simple case in which everyone obtains a most-preferred assignment.

For any agent $x \in N^r$ there exists a non-empty convex set of strictly preferred probability distributions $U_x = \{q \in \Delta^s : u_x(q) > u_x(q_x^*)\}$. Likewise, the set $U = \sum_{x \in N^r} U_x$ is well-defined and convex. Naturally, $U \subset |N^r| \cdot \Delta^s$ (since $\sum_{x \in N^r} q_x = |N^r|$). Let us de-

[4] Agents could be WLOG labeled in a way that the matrices Q^* and V are consistent (i.e. each row refers to the same agent in both matrices).

fine $Y = \prod_{j\in S}[0,\eta_j^r]$, the set of aggregate feasible random assignments, which is also convex. Since Q^* belongs to the weak core (and so does Q^r for E^r) we have $U \cap Y = \varnothing$ (otherwise N^r would be an improving coalition). Applying the separating hyperplane theorem *to the rescaled simplex* $|N^r| \cdot \Delta^s$, there exists a price vector $P \in \mathbb{R}_+^s/\{(p,...,p) : p \geq 0\}$ and a number $w \in \mathbb{R}$ such that $P \cdot a \geq w \geq P \cdot b$, for any $a \in U, b \in Y$. We get rid of price vectors with all equal elements since those would not divide the rescaled simplex in two parts. The object types with excess supply $\left(\sum_{x\in N^r} q_x^{rj} < \eta^{rj}\right)$ would have a zero price component in any such vector P $\left(P^j = 0\right)$.

Let M be a $n \times s$ random assignment matrix (with generic element m_x^j) such that $\sum_{j\in M_x} m_x^j = 1$ for every $x \in N$. Take a random assignment Q such that $diag(VQ') \geq diag(VQ^{*\prime})$. Consider a number $\alpha \in (0,1)$ and build the random assignment $Q^\alpha = \alpha Q + (1-\alpha)M$. Since $q_x^\alpha \in U_x$ for every $x \in N^r$, we have $P \cdot \sum_{x\in N^r} q_x^\alpha \geq w$. Taking the limit, since $\lim\limits_{\alpha\to 1} Q^\alpha = Q$, we have $P \cdot \sum_{x\in N^r} q_x \geq w$.

The same applies to the case $Q = Q^* : P \cdot \sum_{x\in N^r} q_x^* \geq w$. But we know that $\sum_{x\in N^r} q_x^* \in Y$ because Q^* is feasible, implying $P \cdot \sum_{x\in N^r} q_x^* \leq w$. We conclude $P \cdot \sum_{x\in N^r} q_x^* = w$. For, this reason, if we take $q_x \in U_x$ for any agent $x \in N^r$, we have $P \cdot \left(q_x + \sum_{y\in N^r\backslash\{x\}} q_y^*\right) \geq w = P \cdot \left(q_x^* + \sum_{y\in N^r\backslash\{x\}} q_y^*\right)$. Consequently we have $P \cdot q_x \geq P \cdot q_x^*$, proving that P constitutes a pseudomarket quasiequilibrium for this economy E with associated random assignment Q^*.

For each agent $x \in N^r$ such that there exists a probability distribution $\bar{q}_x$ meeting $P \cdot \bar{q}_x < P \cdot q_x^*$, P is indeed a pseudomarket equilibrium price vector. This follows a standard argument. Suppose $q_x \in U_x$ and $P \cdot q_x = P \cdot q_x^*$. Take a number $\alpha \in (0,1)$ and build the random assignment $q_x^\alpha = \alpha \bar{q}_x + (1-\alpha) q_x$, which meets $P \cdot q_x^\alpha < P \cdot q_x^*$. But for α close to 0, $q_x^\alpha \in U_x$, and this would contradict the fact that P constitutes a quasi-equilibrium. Therefore we must have $P \cdot q_x > P \cdot q_x^*$, proving that P constitutes an equilibrium price vector for these agents.

We then focus on the agents for which there is no such probability distribution $\bar{q}_x$. If there is no $q_x \in U_x$ such that $P \cdot q_x = P \cdot q_x^*$, then P is indeed a quasi-equilibrium vector for this agent x. So define $N^c = \{x \in N : \exists q_x \in U_x,\ P \cdot q_x = P \cdot q_x^* = \min\limits_{j\in S} P^j\}$. If $N^c = \varnothing$ we are done since the quasiequilibrium price vector actually constitutes an equilibrium. Thus we assume $N^c \neq \varnothing$.

We claim that our partition starts by setting $N_1 = N \backslash N^c$ (the set for which P is actually an equilibrium price vector with associated random assignment $Q_1^* = [q_x^*]_{x\in N_1}$) and $N_2 \cup ... \cup N_\pi = N^c$. For this we just need to show that N_1 is not empty. If N^M is not empty, we are done. If it is, we know that there exists an "expensive" object type

i such that $P^i > \min_{j \in S} P^j$ (since $P \notin \{(p,...,p) : p \geq 0\}$). If no agent x gets $q_x^{*i} > 0$, then the object type has excess supply implying $P^i = 0$, contradicting $P^i > \min_{j \in S} P^j$. Therefore, some agent $x \in N$ gets $q_x^{*i} > 0$, and consequently $x \notin N^c$. Then $N \backslash N^c \neq \varnothing$ as we wanted to show.

For the next iteration, the "continuation economy" would consist of $S^c = \{j \in S : \eta^j - \sum_{x \in N_1} q_x^{*j} > 0\}$, $\eta^c = (\eta^j - \sum_{x \in N_1} q_x^{*j})_{j \in S^c}$ and N^c. We proceed as in the first iteration to find, subsequently, nonempty disjoint sets $N_2, ..., N_\pi$. For some iteration $\pi \leq n$ we have $N_1 \cup ... \cup N_\pi = N$ since N is finite, and we are done.

Part 2) It follows a recursive argument. Let a blocking coalition $C \subset N$ be defined as follows: $\exists Q$ such that a) $q_x^* \cdot v_x < q_x \cdot v_x$ for all $x \in C$, b) $q_x^* = q_x$ for all $x \in N \backslash C$, and c) $\sum_{x \in C} q_x \leq \eta - \sum_{x \in N \backslash C} q_x^*$. We show that it must be the case that $C = \varnothing$.

We claim that $N_1 \cap C = \varnothing$. If not, there must be a nonempty subset $\tilde{N} \subset N_1$ and an alternative feasible random assignment Q such that $q_x^* \cdot v_x < q_x \cdot v_x$ for all $x \in \tilde{N}$ and $q_x^* = q_x$ for all $x \in N_1 \backslash \tilde{N}$. The SP-equilibrium (with price vector P_1^* associated to N_1) implies $P_1^* \cdot \sum_{x \in N_1} q_x^* < P_1^* \cdot \sum_{x \in N_1} q_x$, and therefore $\sum_{x \in N_1} q_x^{*j} < \sum_{x \in N_1} q_x^j$ for some object type j such that $P_1^{*j} > 0$. Since this price is strictly positive, there is no excess supply in the reduced economy with N_1 on the demand side and η as the supply side. We must have $\sum_{x \in N_1} q_x^{*j} = \eta^j$ and thus $\sum_{x \in N_1} q_x^j > \eta^j$. This constitutes a contradiction as Q is not feasible.

Consequently, $N_1 \cap C = \varnothing$. We focus on the "continuation economy" consisting of $S^c = \{j \in S : \eta^j - \sum_{x \in N_1} q_x^{*j} > 0\}$, $\eta^c = (\eta^j - \sum_{x \in N_1} q_x^{*j})_{j \in S^c}$ and $N \backslash N_1$. Using the same argument in each "continuation economy", we recursively see that $N_2 \cap C = \varnothing$, $N_3 \cap C = \varnothing$... Since $N = \cup_{t=1}^{\pi} N_t$, we conclude that $C = \varnothing$. ■

4 Sequential Pseudomarkets with fixed priorities and budgets: ex-ante Pareto-optimality

We ideally want to fully characterize the set of ex-ante Pareto-optimal random assignments. Since an ex-ante Pareto-optimal random assignment belongs to the weak core, it can be generated by a SP-equilibrium for some ordered partition of the set of agents. Unfortunately, the set of SP-equilibria outcomes may not coincide with the set of ex-ante Pareto-optimal assignments. A simple example with two agents x and y and two objects i and j illustrates this fact. x is indifferent between the objects whereas y strictly prefers object i. If $N_1 = \{x\}$ and $N_2 = \{y\}$ there exists a SP-equilibrium such that x picks i and y picks the remaining object j, which is not Pareto-optimal.

The weak core is a superset of the set of ex-ante Pareto-optimal allocations, therefore the latter concept of efficiency is finer. Nevertheless, we consider that the weak core is a satisfactory concept of ex-ante efficiency because there are no monetary transfers in this economy. A potential coalition that needs to attract an indifferent agent to the coalition in order to make all of its previous members better-off has no means inside a random assignment economy to attract the indifferent agent. However, we also want to explore when the Sequential Pseudomarket can also guarantee an ex-ante Pareto-optimal random assignment.

In order to do so, we take a selection of the family of all SP-equilibrium outcomes in an economy E, which is in principle an uncountable set. Given a partition $\Pi = \{N_1, ..., N_\pi\}$, we endow each agent $x \in N$ with a budget limit $B_x \geq 0$. Let B denote the vector of all budget limits. We select the set of SP-equilibrium assignments where no agent exceeds her budget limit.

We assume hereafter that for no agent there could be two equally valued object types. Then the following assertion holds:

Theorem 2 *Let $\mathcal{Q}$ denote the set of SP-equilibrium random assignments in an economy E with partition Π and budget limits B. Assume that for no agent there could be two equally valued object types. If $\mathcal{Q}$ is countable, then each of its elements is ex-ante Pareto-optimal.*

Proof. Consider $Q^* \in \mathcal{Q}$ not being ex-ante Pareto-optimal, thus some feasible $Q \in \mathcal{F}_E$ ex-ante Pareto-dominates Q^*. Select $t^* = \min\{t \in \{1, ..., \pi\} : Q_{N_t} \neq Q^*_{N_t}\}$. Since Q^* belongs to the weak core, it must be the case that $Q^*_{N_{t^*}}$ is ex-ante Pareto-optimal in the remaining economy E_{t^*} (when we only count the agents in N_{t^*}). Then $q_x \cdot v_x = q^*_x \cdot v_x \ \forall x \in N_{t^*}$.

Let $P^*_{t^*}$ denote the equilibrium price vector for the pseudomarket in a remaining economy E_{t^*} with a set of agents N_{t^*}. Since no agent x is indifferent between any two objects, we cannot have $P^*_t \cdot q_x < P^*_t \cdot q^*_x$ for any $x \in N_{t^*}$, since $q_x \cdot v_x = q^*_x \cdot v_x$. (Else q^*_x would not be an optimal choice with prices $P^*_{t^*}$ and budget B_x: being $M_x^{E_{t^*}}$ x's certain assignment to her most preferred object type that is still available at the remaining economy E_{t^*}, and for $\alpha > 0$ small enough, $\alpha M_x^{E_{t^*}} + (1-\alpha) q_x$ would be a better and affordable choice). Therefore $P^*_{t^*} \cdot q_x \geq P^*_{t^*} \cdot q^*_x$ for all $x \in N_{t^*}$. Consequently we cannot have $P^*_{t^*} \cdot q_x > P^*_{t^*} \cdot q^*_x$ for any $x \in N_{t^*}$. Otherwise we would have $\sum_{x \in N_{t^*}} q_x > \sum_{x \in N_{t^*}} q^*_x$ for some object type i such that $P^{i*}_{t^*} > 0$. Since this price is positive, it must be the case that $\sum_{x \in N_1 \cup ... \cup N_{t^*}} q^*_x = \eta^i$. Hence Q is not feasible, a contradiction.

We conclude that $P^*_{t^*} \cdot q_x = P^*_{t^*} \cdot q^*_x$ for all $x \in N_{t^*}$. For any $\alpha \in [0, 1]$ and any $x \in N_{t^*}$, $\alpha q_x + (1-\alpha) q^*_x$ is also an optimal choice given the budget B_x and prices $P^*_{t^*}$. It is affordable, and it gives the same utility as the optimal choice q_x^*. Then, $\alpha Q_{N_{t^*}} + (1-\alpha)Q^*_{N_{t^*}}$ is a pseudomarket equilibrium random assignment for E_{t^*} with same prices and budgets. So every $Q^\alpha = [Q^*_{N_1}, ..., Q^*_{N_{t^*-1}}, \alpha Q_{N_{t^*}} + (1-\alpha)Q^*_{N_{t^*}}, Q^\alpha_{N_{t^*+1}}, ..., Q^\alpha_{N_\pi}] \in \mathcal{Q}$, where $Q^\alpha_{N_{t^*+1}}, ..., Q^\alpha_{N_\pi}$ are pseudomarket equilibrium random assignments of the subsequent remaining economies. (These subsequent equilibria exist by Hylland and Zeckhauser, 1979). But this is in contradiction with $\mathcal{Q}$ being countable. ■

The conclusion is that when we fix a priority structure and a profile of budget limits, SP attains ex-ante Pareto-optimality almost certainly. Two events could preclude this assertion, yet both are generically not to be expected. One is that indifferences between object types arise.[5] The other one is that a remaining economy is not regular, giving a continuum of pseudomarket equilibrium outcomes.

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[5] In that case, SP equilibria could be refined in the sense of selecting a cheapest bundle among all optimal affordable choices. Then we could prove that the theorem holds even under indifferences. However, existence of a SP equilibrium meeting this refinement requires a proof beyond Hylland and Zeckhauser's (1979) technique.