School Choice Mechanisms, Peer Effects and Sorting

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December 19, 2014

Abstract

We study the effects that school choice mechanisms and school priorities have on the degree of sorting of students across schools and neighborhoods, when school quality is endogenously determined by the peer group. Using a model with income or ability heterogeneity, we compare the popular Deferred Acceptance (DA) and Boston (BM) mechanisms under several scenarios. With residential priorities, students and their households fully segregate into quality-ranked schools and neighborhoods under both mechanisms. With no residential priorities and a bad public school, DA does not generate sorting in general, while BM does so between a priori good public schools. With private schools, the best public school becomes more elitist under BM.

Key-words: School choice, mechanism design, peer effects, local public goods.
JEL classification numbers: I21, H4, D78.

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1 Introduction

School choice is one of the most hotly debated and rapidly expanding education policies.[1] Advocates (e.g. Friedman, 1955; Hoxby, 2003) claim that school choice policies could allow equal access to higher quality schooling for all, and so be a tide that lifts all boats. On the one hand, it is argued, choice introduces competition into the education system, pushing schools to be more productive.[2] On the other hand, affluent families always had choice, as they could afford private schooling or housing in expensive areas, and so introducing choice should improve equity by allowing poor parents to choose as well. Critics (e.g. Smith and Meier, 1995; Musset, 2012; OECD, 2012) retort that school choice may instead exacerbate educational inequality and harm vulnerable students by increasing socioeconomic and ability segregation across schools and leaving them behind in lower quality schools. The main reasons are that more educated parents make better informed choices, and that low income households have their effective choice sets restricted as they cannot afford transport and other direct or indirect monetary costs.

This paper contributes to the debate by examining whether the application of two widely used school choice mechanisms, Deferred Acceptance (DA) and the Boston mechanism (BM),[3] can result in socioeconomic sorting.[4] So as to stack the deck against the emergence of segregation, we study a model without transport costs and where all parents are rational and posses the same information. Even so, our results prove that DA and BM may well generate socioeconomic sorting across schools and residential areas and that, depending on the details, the strategic differences between them sometimes lead to sharp differences in the distribution of students across schools.

Our contribution builds a bridge between two important and largely dis-

[1] For instance, more than two thirds of OECD countries have expanded school choice opportunities in the last decades (Musset, 2012).

[2] That claim is not free of controversy and the question has not yet been settled in the literature. Several theoretical contributions explain why school competition may *harm* school productivity in the presence of reputation effects or asymmetric information (De Fraja and Landeras, 2004; MacMillan, 2004, MacLeod and Urquiola, 2008). On the other hand, the empirical evidence is at best inconclusive (Hoxby, 2000, 2003, 2007; Rothstein, 2007; Gibbons et al., 2010, OECD, 2014).

[3] Most of the mechanism design literature on school choice has debated the properties of BM and DA. BM is used in places such as Denver or Barcelona, while DA is used for example in Boston or New York City.

[4] We use the terms sorting and segregation interchangeably.

connected literatures in the economics of education. A first strand, inspired by Tiebout (1956), treats schooling as a local public good, assuming that children are assigned to their local school.[5] The choice of where to live embeds the choice of school and, since better-off households are willing to pay more for school quality, socioeconomic segregation across communities and their schools ensues.[6] In that setting, frictionless school choice (i.e. with neither transport costs nor capacity constraints) indeed prevents segregation in the public sector (Epple and Romano, 1998, 2003): schools must admit any applicant and thus must be of homogenous quality in equilibrium, since otherwise those assigned to a bad school could be better-off by applying to a better one.[7] Epple and Romano (2003) conjecture that their results would extend to a model where public schools had limited capacity and overdemands were resolved through lotteries, but do not provide relevant details of the school choice mechanism used (e.g. what happens with children excluded from their first choice). Those details and the properties of the resulting match of children to schools are precisely the focus of the second literature this paper belongs to.[8]

Started by Abdulkadiroglu and Sönmez (2003), that literature reveals the importance of the rules applied to resolve overdemands when limited school

[5]Tiebout (1956) presented his seminal contribution as a response to Musgrave's (1939) and Samuelson's (1954) result that "no "market-type" solution exists to determine the level of expenditures on public goods" . He suggested that a solution to the preference revelation (or free-rider) problem could exist for local public goods –i.e. those only available to the residents of a local community and so excludable. Tiebout's path-breaking basic idea is that households living in metropolitan areas where different communities offer different combinations of local public good provision and taxation, "shop" for local public goods by choosing where to live and thus reveal their preferences.

[6]In early contributions, school quality differences emerge and sustain segregation because wealthier communities, with a larger tax base, vote for larger levels of public spending in education (e.g. Epple et al., 1984). In more recent ones, the peer group effect explains the emergence of such differences across the schools of a single district (e.g. Bénabou, 1996; Epple and Romano, 2003; De Fraja and Martinez-Mora, 2014).

[7]Epple and Romano (2003) also study the effects of transport costs and find that full residential segregation and partial school segregation by income would emerge in equilibrium.

[8]Important contributions to this literature include Bénabou (1993), which reveals a way whereby socioeconomic segregation may create poverty traps and ghettos; Durlauf (1996), which explains how socioeconomic segregation can perpetuate income inequality across generations; and Nechyba (2000), who shows how the existence of private schools may reduce socioeconomic segregation by severing the link between a household place of residence and the school the child attends.

capacities preclude the immediate satisfaction of parents' first choices. It formally analyzes the game generated by a centralized system where families submit a ranking of schools and a set of rules determines who gets accepted in an overdemanded school and what options are left for rejected applicants. These rules define the so-called school choice mechanisms, which often include priorities for applicants living in the neighborhood of the school or having a sibling in the school. Abdulkadiroglu and Sönmez (2003), and a fruitful literature derived from it, define several properties that these mechanisms should satisfy and establish a tradeoff between efficiency (satisfying parents preferences) and stability (respecting priorities).[9]

We propose a unified theoretical framework that merges these two literatures. It is, to the best of our knowledge, the first one to embed the school choice problem in a multi-community model of local public good provision. Our framework allows us to gain a better understanding of the impact school choice design has on socioeconomic sorting into schools and neighborhoods.

Our base model represents a city divided into three districts with a continuum of households that differ in a unidimensional type, the minimal structure we need to present our results.[10] Every household has a parent and a school-aged child. We will interpret household type either as parental income or the child's ability, depending of the form of the utility function.[11] Parents first decide which district to live in and then participate in a school choice mechanism that assigns their child to a public school. Finally, if private schools are available, they choose whether to keep the child in the public system or to pay for private schooling. School quality is a function of the characteristics of the student body, summarized by the average peer type.[12] Household's

[9]This tradeoff has been argued to be small –Chen and Sönmez (2006) question its relevance through lab experiments.

[10]This is only for mathematical simplicity. We show in the appendix that the main ideas and results of the paper extend naturally to more general specifications.

[11]The robustness analysis presented in the Appendix proves that our results hold with a different characterization of exogenous quality differences and with an arbitrary number of districts and schools. Moreover, it contains an extension of the model in which households differ along two dimensions: parental income and child ability. Qualitative results do not change.

[12]In line with Epple and Romano (2011), we define peer effects as any influence that a student has in the learning of her class or school mates. There is a large and growing body of literature studying the empirical relevance of peer effects and the mechanisms through which they affect the educational process. A consensus exists that they matter, and that a "better" peer group enhances performance (Epple and Romano, 2011; Sacerdote, 2011).

utility depends on current consumption and the future human capital of the child. Overall, our approach is the first to endogenize both priorities, by allowing families to choose where to live, and preferences, by making school quality a function of the peer quality of the school's student population.[13]

A first important observation of our analysis is that residential priorities (or any that parents can somehow pay for) prevail over the mechanism itself when schools have priorities for local applicants, perfect income segregation across districts and schools characterizes equilibrium, as if the place of residence directly determined where children go to school. The take-home message is that policy-makers should not expect to reduce school segregation by implementing choice and having tradable priorities break ties, as it is the case in most countries with active school choice policies.

When schools have no priorities, the mechanisms do not induce sorting into neighborhoods, but may lead to starkly different distributions of students across schools. If there are no bad schools (loosely defined as schools which parents would like to avoid ex-ante), no segregation across public schools will emerge. However, if bad schools exist or parents think they do (e.g. for being placed in a ghetto or at the bottom of the local league table), BM can induce *ability segregation* across *good* public schools (hence "elitizing" one of the good schools) if the child's ability and school quality are complements in the production of human capital. In sharp contrast, DA induces homogeneous quality across public schools and no segregation.

We next show that the presence of private schools has a profound effect on the allocation of children across public schools. In this scenario, segregation by ability is larger in the BM, reinforcing the "elitization" of the best public schools where top types obtain easier access. Furthermore, the BM can also have an equilibrium with *income segregation* across public schools. Therefore, if the BM is used, the existence of a bad school can trigger the same *cream-skimming effect* previously found for private schools (see Epple and Romano, 1998 and Epple et al., 2004) *within the public sector*. On the other hand, in DA, exogenous quality differences among public schools are exacerbated by the peer effect, since private schools attract more good students from schools with lower exogenous quality. Hence, the only way to fully avoid the emergence of segregation in the system with perfect information and no

[13] On-going research by Estelle Cantillon (2014) studies how group admission quotas can avoid the emergence of segregation when preferences are endogenously determined by peer quality.

transport costs is to have DA without either outside options or residence priorities.[14]

To sum up, our results suggest that school choice must be carefully designed if socioeconomic segregation is to be avoided. Its essence is to introduce and promote competition between schools (e.g. through the publication of league tables), while ghetto schools are usually present, even in OECD countries. Recent work by Calsamiglia et al. (2014) and Calsamiglia and Güell (2014) provides empirical evidence showing that priorities play a large role in the final allocation of students to schools when residential priorities exist.[15] Although they do not consider peer effects or residential choices explicitly, the sorting effects that we identify in BM seem empirically plausible in the light of their results. Similarly, these papers find that a substantial fraction of families taking risks in the city of Barcelona opt for a private school if they do not get the desired school, empirically validating the channel that private schools play when BM is used.

Section 2 presents the basic elements of the model, the main assumptions and the equilibrium notions, while section 3 provides a detailed explanation of both mechanisms. Section 4 starts the analysis in a setup where schools have priorities for students living in the neighborhood. We then, in Section 5, study scenarios where schools have no priorities: we focus on the case without private schools in 5.1, and then extend the analysis by introducing private alternatives in 5.2. The final section concludes the paper while discussing efficiency considerations. All the proofs as well as the robustness analysis are gathered in the Appendix.

2 The model

The model represents a city divided into three equally sized school districts with fixed boundaries. Districts and their schools are indexed with $j = 1, 2, 3$. Each district has a school that offers tuition-free public education. A population of households with mass normalized to 1 lives in the city. Every household consists of a parent, who takes decisions, and a school-aged child.

[14]This is the public sector counterpart to the vouchers system proposed by Epple and Romano (2008) to avoid segregation with private schools. In their proposal, private schools can select students and optimal vouchers compensate for externalities so that schools are indifferent among all students: equal quality ensues.

[15]That is the case in most OECD countries (OECD, 2012).

Households (sometimes called agents, or students) differ continuously along a single dimension. Household type is denoted with $t \in D \equiv [\underline{t}, \bar{t}]$, and is distributed in the population according to a continuous and strictly increasing distribution function $\Phi(t) \in [\underline{t}, \bar{t}]$. We denote with $t_{(\phi)}$ the ϕ quantile of Φ, i.e. $\Phi(t_{(\phi)}) = \phi$. We interpret types either as household income (or human capital) or as the child's ability (or school readiness), depending on the form of the household utility function.

Housing market. Districts have a fixed supply of homogeneous houses owned by absentee landlords. Households must rent a house to live in. The rental price of a house in district j is denoted by r_j. In order to avoid an uninformative multiplicity of equilibria, we anchor housing prices by normalizing the lowest rental price to zero.

Households have identical **preferences** over combinations of (current) consumption of a private good x –the numeraire– and the child's (future) human capital h.[16] A separable utility function represents these preferences: $U(x, h) = u(x) + h$.

The amount of **human capital** accumulated by the child at the end of the education process is a function of the quality of schooling received q and of the household type t: a child from a $t-$type household who attends school j derives human capital $h(q_j, t)$. u and h are twice differentiable[17] and increasing. A parent of type t who pays rent r for the house consumes $x = t - r$; hence the indirect utility function is $V(t, q, r) = u(t - r) + h(q, t)$.

Two sources of variation in the demand for school quality may trigger the emergence of segregation in our model: on the one hand, if school quality and type are complements in the production of human capital, higher types will benefit more from school quality; on the other, if the marginal utility of income is decreasing, higher types will be willing to pay more for school quality.[18] In order to identify the relevant sources of segregation in each scenario in a transparent way, we will be using one of the following assumptions as we go along:

[16] Houses are homogeneous and so excluded from the preference relation.

[17] Differentiability is actually not necessary to obtain our results: we just require U to be continuous and monotone.

[18] Formally, agents' demand for school quality increases with type if V_q/V_t does so, which requires either $h_{qt} > 0$ or $u'' < 0$.

Assumption 1 (A1) *Marginal utility of income is constant:* $u'' = 0$*; school quality and type are complements in the production of human capital:* $h_{qt} > 0$*. Moreover, the lowest type* $\underline{t}$ *has no ability to benefit from better school quality, so that* $h(\cdot, \underline{t})$ *is a (weakly) positive constant function.*

Assumption 2 (A2) *Marginal utility of income is decreasing:* $u'' < 0$ *and* $u(0) = -\infty$*; school quality and type are independent in the production of human capital:* $h_{qt} = 0$*.*

Under assumption A1, our model is best interpreted as one where agents differ by ability, while under assumption A2 our type can be best thought of as parental income or human capital.

School quality depends on the distribution of types at the school.[19] To simplify the exposition, we follow the bulk of the theoretical literature and assume that school quality is given by the average type of the peer group (e.g. de Bartolomé, 1990; Epple and Romano 1998; De Fraja and Martínez-Mora, 2014)[20] and so, denoting with Φ_j the distribution of students' types conditional on being assigned to school j, we have:[21]

$$q_j = \mathbb{E}_{\Phi_j} t.$$

[19] We do not contend that other inputs such as spending per pupil and teacher quality are important in the education production process. However, ours is a model of interaction between schools ascribed to the same educational authority. Therefore, we consider that other inputs are equally distributed across schools and assume them away.

[20] This means we assume school quality increases with the average income or human capital of parents when we interpret type as income. This is in line with results in De Fraja et al. (2010). Among other results, they obtain evidence that: (i) the effort of parents and children at school are strategic complements; (ii) parents with more education, higher income and higher socioeconomic status exert more effort in their children's education (so that the children work harder at school), while those that endure financial hardship exert less effort; (iii) schools exert greater effort when they have a larger proportion of students from better-off households; and (iv) effort strongly improves school achievement.

[21] Our results extend to other, more general, formulations of the peer group effect. For example, they hold in a setting where a smaller proportion of children with ability below a certain threshold and a larger proportion of children with ability above a larger threshold enhance performance, as in Summers and Wolfe (1977). Qualitative results extend as well to a setting where quality is affected by the degree of heterogeneity in abilities at the school, as in Bénabou (1996a).

When we study the impact of **failing schools**, we will also assume that there is an observably *bad school* which WLOG we label school 3.[22] We model this by assuming that being assigned to school 3 induces a fixed reduction of $\Delta > 0$ units in the future human capital of the child. We sometimes call this the *ghetto effect*.[23]

School capacity is assumed to be identical to the number of children residing in the district, and so it is equal to $\frac{1}{3}$ for all schools.

We next provide a few formal definitions before discussing the timing of the model and the notions of equilibrium we use. Let $H : D \to \Delta(\{1, 2, 3\})$ be the function representing the (possibly random) choice $H(t)$ of residential district of each type t, Ξ denote the set of all measurable housing choice profiles H that clear the housing market (i.e. such that H is feasible). Let $R : D \times \Xi \to \mathcal{P}(\{s_1, s_2, s_3\})$ be the function representing the ranking strategy $R(t, H)$ of household t, given the housing choice profile $H \in \Xi$;[24] $\mathcal{P}$ denotes the set of all possible rankings (permutations) among the three schools.

Timing of the model

1. Households first choose which district to live in, $H(t) \in \Delta(\{1, 2, 3\})$, and rent a house there.

2. The school-choice mechanism M is then applied: households submit a ranking (i.e. a linear ordering) of the three schools, $R \in \mathcal{P}(\{s_1, s_2, s_3\})$, and the rules specified in M select an assignment of children to schools. The allocation of children to schools in turn determines the peer groups and the quality of schools. Finally, payoffs are realized.

3. Finally, if a private school is available, parents choose between the public school the child is assigned to and the private school.

[22] We argue that the existence of public schools with a bad reputation and which parents wish to avoid is prevalent across the world. Most cities have deprived areas or slums where the poor population concentrates and where, arguably, schools have a self-fulfilling bad reputation, at least, among non-residents. This is further facilitated in countries that publish school league tables.

[23] We consider an alternative modelling of such bad school in the appendix. In the alternative modelling, school 3 quality is discounted by a factor $\delta < 1$. Results are qualitatively identical.

[24] We focus on pure ranking strategies since Mas-Colell (1984) guarantees the existence of a Nash equilibrium in pure strategies at this stage of the game for a continuum of households.

Equilibrium concepts

An **equilibrium** given a mechanism M is a tuple of *beliefs* about qualities $(\hat{q_1}, \hat{q_2}, \hat{q_3})$, housing rents (r_1, r_2, r_3), a housing choice profile $H^* \in \Xi$ and a ranking strategy profile $R^*(t, H)$ for each $t \in D$ and $H \in \Xi$ such that

1. *Rational choices*: Given the beliefs $(\hat{q_1}, \hat{q_2}, \hat{q_3})$, no household can increase utility by changing their choice of residence or their ranking of schools.

2. *Consistent beliefs*: Given the assignment provided by M, induced qualities coincide with the believed qualities: $\hat{q}_j = q_j \forall j$.

3. *Housing markets clearance*: the demand and supply of houses balances in the three districts.

We define a **sequential equilibrium** given a mechanism M as an equilibrium such that there exists a sequence of beliefs $(q_1^n, q_2^n, q_3^n)_{n\in\mathbb{N}} \rightarrow (q_1, q_2, q_3)$ satisfying $q_i^n/q_j^n \neq q_i/q_j$ for all $i, j \in \{1, 2, 3\}$ and $n \in \mathbb{N}$,[25] a housing choice profile sequence $H^n(t)$ and a sequence of ranking strategy profiles $R^n(t, H)$ such that

1. For each type t, the pair $(H^n(t),\ R^n(t, H))$ is best responding to R^* and H^* given (q_1^n, q_2^n, q_3^n) and (r_1, r_2, r_3)

2. Given the assignment produced by M, $(H^n,\ R^n) \rightarrow (H^*,\ R^*)$ as $n \rightarrow \infty$.

Definitions of sorting. Our definitions of sorting are based on comparisons of the ex-post distribution of types between pairs of schools and neighborhoods. We say that there is **full sorting** between schools i and j if $\sup(supp(\Phi_j)) \leq \inf(supp(\Phi_i))$. That is, the maximum type assigned to school j lies weakly below the minimum type assigned to i. There is **partial sorting** between schools i and j if $\Phi_i FOSD \Phi_j$, implying $q_i > q_j$.[26] There

[25] We tremble the ratio between qualities instead of the qualities themselves because we want each pairwise school comparison in the sequence to differ from the equilibrium pairwise comparison.

[26] Other forms of sorting could also be explored. For instance, sorting coming from second-order stochastic dominance. In this paper, however, our results concern the kinds of sorting defined in the main text and so we do not provide other definitions of sorting.

is **no sorting** between schools i and j if $\Phi_i = \Phi_j$. Analogous definitions of sorting into residential areas establish whether the neighborhoods of the city display full, partial or no socioeconomic sorting at all.

3 The assignment mechanisms

Before we proceed with the analysis, we briefly explain how the two assignment mechanisms we study, BM and DA, assign students to schools. In both mechanisms, parents are requested to report a complete ranking of the available schools to the school authority. Both mechanisms use an algorithm that assign children to schools round by round. In the first round, each student is considered for the school her parents ranked first. If the number of students considered for a school exceed the capacity of that school, some students will need to be rejected, following the school priorities and a tie-breaking lottery when necessary. Each student rejected in some round goes to the next round where she is considered for the highest-ranked school that has not rejected her yet. The difference between BM and DA stems from how these mechanisms treat the students that are not rejected (i.e. the accepted students). In BM, every accepted student keeps her slot at the school for which she was considered and both the student and the slot are removed from the assignment algorithm (definite acceptance). In DA, an accepted student only gains the right to be reconsidered for the same school in the next round, meaning that rejection in posterior rounds is possible (deferred acceptance indeed).

While the way BM proceeds is easier to understand for parents, DA has the advantage of strategy-proofness (as shown by Roth, 1985, Theorem 5). An assignment mechanism is strategy-proof if providing truthful information about one's own preferences when asked constitutes a weakly dominant strategy (i.e. it is always a best response to any profile of the other agents' strategies). In school choice problems, this property provides a valuable simplification of the strategy choice parents face, since they cannot do better than report their true ordinal preferences. This may not be the case in BM. Given that slots are definitely given round by round, the opportunity cost of truthfully reporting preferences is the reduction of available slots in not-so-preferred schools in further rounds. Thus, each parent needs to balance her preferences with her chances. Consequently, she may rank a moderately good school with high acceptance chances in first position, and so on.

In an environment with peer effects, parents' preferences are also affected by the average school type, an endogenous outcome. Thus, strategy-proofness is not a guarantee of strategic simplicity, since each parent still needs to take other parents' strategies into account in order to construct her own preferences. The value of strategy-proofness is then diminished. On the other hand, BM, precisely because it is not strategy-proof, manages more information about parents' preference intensities than DA does. In fact, parents with the same ordinal preferences may report different rankings if their preference intensities are different. This feature yields some efficiency properties for BM that DA usually does not attain (Miralles, 2008; Abdulkadiroglu, Che and Yasuda, 2011). This paper studies and compares the degree of students and households sorting into schools and residential areas that DA and BM may generate.

4 School choice with residential priorities

In this section we consider a setting where residents of a district have priority over non-residents when competing for slots in the local public school. An important result is that the school choice mechanism plays no role in this context. The choice of school is effectively embedded in the choice of neighborhood and both mechanisms lead to exactly the same outcome with perfect sorting across schools and neighborhoods.

We solve the game by backward induction. An important observation is in Lemma 1, which shows that the two mechanisms result in all children attending their own neighborhood school; then, proposition 1 proves that perfect sorting across neighborhoods and their schools emerges in the unique equilibrium with differentiated qualities.

Lemma 1 *Consider any house choice profile $H(t) \in \Xi$ such that districts differ in their demographic composition. Then both BM and DA lead to the same and unique equilibrium of the assignment stage in which students with types $H(t) = j$ attend the school of district j.*

Proposition 1 *Suppose A1 or A2 hold. Then if schools have residential priorities, both DA and BM result in a unique equilibrium with $q_1^* > q_2^* > q_3^*$ such that $H^*(t) = 1$ if $t \in (t_{(2/3)}, \bar{t}]$, $H^*(t) = 2$ if $t \in [t_{(1/3)}, t_{(2/3)}]$ and $H^*(t) = 3$ if $[0, t_{(1/3)})$. Equilibrium rents satisfy $r_3 = 0 < r_2 < r_1$. There is full sorting of households between all pairs of schools and neighborhoods. This equilibrium is sequential.*

The proof is straightforward and thus we omit it. Simply note that, under A1 or A2, the willingness to pay for school quality is increasing in t, either because higher types benefit more from school quality (A1), or because they have a smaller marginal utility of income (A2). It is usual in Tiebout-type models to also have symmetric equilibria where all schools and residential areas, as well as house prices, are identical. Such equilibria are however typically unstable. The next questions we ask are therefore whether our model has an equilibrium without sorting when schools have residential priorities and whether it is sequential.

Remark 1 *There is an equilibrium where $q_1 = q_2 = q_3 = \mathbb{E}(t)$. Types evenly randomize between renting at each district, and so $r_1 = r_2 = r_3 = 0$; at the assignment stage of the game, all households rank their local school first. However, this equilibrium is not sequential. Consider any sequence of beliefs summarized by $(q_1^n, q_2^n, q_3^n)_{n=1,2...} \to (q_1, q_2, q_3)$ such that $q_1^n \neq q_2^n \neq q_3^n$. For each n, housing demand will be concentrated on the district whose school has higher quality so that housing demand does not converge to the even randomization between districts.*

The emergence of full residential and school segregation when schools have residential priorities is a prediction in line with Bénabou (1996) and Epple and Romano (2003). It is important to point out that this is not a phenomenon arising only under the existence of residential priorities. *Anything* that can generate a monetary market for priorities readily does so.

5 School choice with no priorities

In the rest of the analysis, we consider setups where the place of residence does not carry any priority in the school assignment procedure.[27] An immediate implication for both DA and BM is that the housing market does not play any role in those setups.[28]

Lemma 2 *Suppose there are no priorities. Then, equilibrium in the housing market has $r_1 = r_2 = r_3 = 0$ and is compatible with any allocation of agents to*

[27] The results exposed in this section could be immediately generalized to the case of having an arbitrary number of schools.

[28] Note that for that result to be true, it is essential that no additional elements (e.g. transport costs) are included.

districts that clears the housing markets; therefore no residential segregation emerges.

We omit the easy proof but note that for lemma 2 and the following proposition to hold it is also essential that no transport costs are included. The next result proves that with ex-ante symmetric schools and without priorities neither BM nor DA engender any socioeconomic sorting.

Proposition 2 *Suppose there are no priorities and that public schools are ex-ante identical. Then, there is no equilibrium in DA or BM with differentiated school qualities (i.e. such that for some pair of schools i and j we have $q_i \neq q_j$).*

Instead, equilibrium in both cases has homogenous schools.

Remark 2 *Suppose there are no priorities and that schools are ex-ante identical. Then, there is an equilibrium in DA and BM without school sorting: Indeed, if agents believe $q_1 = q_2 = q_3$, they all may play the same ranking strategy, say putting school 1 first, school 2 second and school 3 third. Consequently, we would have $q_1 = q_2 = q_3$ ex post. Moreover, this equilibrium is sequential. For DA one can take any sequence of beliefs $(q_1^n, q_2^n, q_3^n)_{n=1,2...} \to (q_1, q_2, q_3)$ with $q_1^n > q_2^n > q_3^n$ such that the best response profile always consists of everyone putting school 1 first and putting school 2 second. For BM, it is key that q_2^n and q_3^n converge to equality much faster than q_1^n and q_2^n do. The risk of applying for school 1, which is going to be overdemanded, in first place, instead of applying for the safer option (school 2) is the risk of ending up in school 3 with higher probability. Given the converge speeds, this risk is relatively negligible as compared to the premium between the best school and the second-best school. Asymptotically almost every type puts school 1 in the first position of submitted ranking (and school 2 in second position - school 3 is the worst school and ranking it other than last is part of a dominated strategy). Consequently $(q_1^n, q_2^n, q_3^n)_{n=1,2...} \to (q_1, q_2, q_3)$.*

5.1 School Choice with failing schools

In this subsection, we introduce some ex-ante heterogeneity in schools. In particular, we assume that one of the schools is of lower quality than the rest for some exogenous reason (e.g. for being located in a bad neighborhood), independently of the distribution of children across schools. We refer to it as the ghetto school[29] and assume WLOG that it is school 3. Formally, human capital if school 3 is attended is $h(q_3, t) - \Delta$, where Δ is the cost, in terms of future human capital, of attending the ghetto school. When we study the BM we will assume $\Delta > \Delta^* \equiv h(q_{\max}, \bar{t}) - h(q_{\min}, \bar{t})$, where $q_{\min} \equiv \mathbb{E}(t \left| t < t_{(1/3)} \right.)$ (the minimum possible school quality) and $q_{\max} \equiv \mathbb{E}(t \left| t > t_{(2/3)} \right.)$ (the maximum school quality that can be attained in equilibrium). That is, the ghetto effect dominates potential quality differences due to peer effects, implying that school 3 is always last in households' preferences.

With ex-ante heterogeneity in the quality of schools, the strategic differences between the two mechanisms come into play. We find that, even if schools have no priorities, the existence of a failing school may well generate school sorting in the BM, but not in DA.[30]

Proposition 3
a) Suppose A1 holds. Then, for any $\Delta > \Delta^*$, there is an equilibrium in BM with neither priorities nor private schools with a strategy profile characterized by a threshold $\tilde{t} \in (\underline{t}, t_{(1/2)})$ such that all types above the threshold rank school 1 first and all types below it rank school 2 first. School 3 is ranked last by every type. If $\Phi(\tilde{t}) \geq 1/3$ this equilibrium brings full segregation between schools 1 and 2. Segregation is partial if $\Phi(\tilde{t}) < 1/3$. Moreover, this equilibrium is sequential. If Δ is high enough, this sequential equilibrium is unique and entails full sorting.[31]
b) For any $\Delta > 0$, and under A1, there is a sequential equilibrium under the DA mechanism that displays no sorting and has qualities $q_1 = q_2 = q_3$.

In the BM, parents need to think strategically: since assignments are final in each round, ranking a high quality and overdemanded school first increases the chances of not getting a slot in the first round, and reduces the probability of being admitted in a not-so-preferred, but still acceptable, school in further rounds. Therefore, truth-telling has an opportunity cost: the higher risk of having their child assigned to the ghetto school. Parents must balance their preferences with their chances.

[29] However, the interpretation is broader: we only need a school that all parents wish to avoid, for instance because it is placed at the bottom of a school league table.

[30] Note that, as in the previous section, equilibrium in the housing market does not display any kind of sorting across districts, that is, the rent is zero in every district (lemma 2).

[31] The same result can be shown even for $\Delta > h(\mathbb{E}(t \left| t > t_{(1/2)} \right.), \bar{t}) - h(q_{\min}, \bar{t})$.

Under A1, lower types benefit from (and so value) school quality less than higher types. In an equilibrium with segregation, low type parents misreport their preferences (ranking school 2 first) in order to reduce the chances of having their child exposed to the ghetto effect. Parents of children of higher ability prefer to take the risk and rank the best school first. Hence, the existence of a bad school (school 3) generates sorting between other ex-ante identical schools (schools 1 and 2).[32] Note that diminishing marginal utility of income (A2) is neither sufficient nor necessary for a segregation equilibrium to exist in BM in this scenario, since public schools are free and there are no priorities that parents can pay for.

Deferred Acceptance does not create strategic differences between those who value peer quality more and those who value it less. Because there are no gains to be made by a household who misreports their preferences, all parents submit the same true ranking of schools and have the same probability of being assigned to each school. Hence no differences emerge in the demographic composition of schools.

Remark 3 *Interestingly, in a segregation equilibrium in BM, the bad school has better ex-post peer quality than the second best public school ($\Phi_3 FOSD \Phi_2$), which partially compensates the ghetto effect. Moreover, the ex-post peer quality of school 3 under BM exceeds $\mathbb{E}t$, the ex-post quality of schools under DA. Both results easily fade away if private schools are available (see below).*

5.2 School choice with failing and private schools

This subsection introduces private schools into our base model to investigate the way outside options affect the behavior of parents and the resulting allocation of children across public schools. Outside options are typically available in school markets. However, to the best of our knowledge, we are

[32] The BM also has an equilibrium with no school sorting, just as in DA, but this equilibrium is not sequential. If all agents believe that in equilibrium school qualities are going to be identical, there will be an equilibrium in which agents choose their strategies (say, strategy 1 with probability 1/2 and strategy 2 with probability 1/2) in a way that the average quality among the finally assigned students remain equal between schools 1 and 2 ex post ($q_1 = q_2 = \mathbb{E}t$). However, consider any sequence of beliefs summarized by $(q_1^n, q_2^n)_{n=1,2...} \to (\mathbb{E}t, \mathbb{E}t)$ such that $q_1^n \neq q_2^n$. For each n, the profile of best responses can be characterized by a threshold $\hat{t}^n$ separating types who play strategy 1 from types who play strategy 2. Then the equilibrium with mixed ranking strategies best does not arise as we approach the limit of that sequence.

first in studying the workings of school choice *mechanisms* in the presence of both private schools and peer effects.

For simplicity, we assume there is a single private school and that its price is exogenous but our results can be generalized to more nuanced models of the private sector. Parents now have a choice between the public school the child is assigned to and a private school of quality q_p, capacity η_p and price $p > 0$ *after* the school assignment algorithm is completed. The private school uses the same technology to produce education and so its ex-post quality q_p is given by the mean ability of its student body.

Note that, by lemma 2, households are indifferent about where to live and any allocation of households to districts that clear the housing markets is an equilibrium of the location stage. The choice between public and private schooling at the last stage of the game does not depend on the details of the mechanism previously used. Therefore, the following single-crossing condition holds both with DA and BM. Let t_j be the household type that is indifferent between school j and the private school (note that it depends on p, q_p and q_j).

Lemma 3 *Suppose A1 or A2 hold. If $q_j < q_p$ and the price of the private school $p > 0$: if a household with a child of ability t prefers the private school to school j, then so do all households with children of ability $t' > t$; and if a household with a child of ability t prefers school j over the private alternative so do all households with children of ability $t' < t$.*

5.2.1 Deferred Acceptance

In the presence of a private alternative, DA may well generate partial segregation across public schools, provided demand for school quality increases with type. If public schools are heterogenous ex-ante, those of higher exogenous quality are able to retain more high type students. The consequence is that (endogenous) peer quality differences emerge across public schools, reinforcing the initial (exogenous) differences. This equilibrium displays a hierarchy of school qualities and partial sorting but, because all agents submit the same ranking of preferences and schools have no residential priorities, the probabilities of admission at each public school are identical for all types.[33]

[33] Proposition 4 adopts assumption A1; it is straightforward to derive an analogous result with preferences that satisfy A2.

Proposition 4 *Suppose A1 holds and let $p > \Delta > 0$. Then, there is an equilibrium in DA with no sorting between schools 1 and 2, but with partial sorting between them and school 3 provided $p < \Delta + h(\bar{t}, \bar{t}) - h(\mathbb{E}t, \bar{t})$.*[34]

Proposition 4 imposes an upper bound on the price of the private school which guarantees that $t_3 < \bar{t}$, so that at least some households acquire private education. It is worth noting that the result does not require the ghetto effect to dominate potential quality differences due to peer effects (as the emergence of sorting requires in BM). Nevertheless, it is possible to show that the stronger the ghetto effect (i.e. the larger the initial differences across public schools) the larger the final quality gap between the bad and the good public schools.[35]

5.2.2 Boston Mechanism

The availability of a private school also affects the outcome of the BM in important ways. On the one hand, the complementarity between school quality and type is no longer necessary for the emergence of segregation when agents have an outside option: an equilibrium displaying *income segregation* exists if the marginal utility of income is decreasing. On the other hand, if school quality and the child's type are complements in the production of human capital, the availability of a private alternative may well lead to an equilibrium with a more elitist best public school. This gives rise to a new source of unfairness: *top types have higher chances of admission at the best public school when a private school is present.*

We start with the latter result. Let $\tilde{t}_{\max}$ denote the maximum equilibrium cut-off in the BM game with neither priorities nor private schools. We look for an analogous equilibrium cut-off $\tilde{t}^p$ when we include a private school with price p. Such cut-off exists since *single-crossing conditions* apply: if a type t best responds by ranking school 1 in first position, a higher type must also do so. Likewise, if a type t ranks school 2 in first position, a lower type must do so as well. We prove in proposition 5 that this is indeed the case.

[34] Notice that proposition 4 does not rule out the possibility that some households assigned to schools 1 or 2 opt out as well. Indeed, that could happen if $p < h(\bar{t}, \bar{t}) - h(\mathbb{E}t, \bar{t}) < \Delta$. Of course, since $\Delta > 0$, in that case the cut-off type would be larger in schools 1 and 2 than in school 3 and the result would still hold.

[35] In cases where $p < \Delta$, the lowest type prefers the private alternative over school 3 and, by lemma 3, so do all agents. But higher types would then segregate from lower ones by staying in school 3 provided $\Delta - p < h(\bar{t}, \bar{t}) - h(\mathbb{E}t, \bar{t})$. See footnote 39.

Proposition 5 *Suppose A1 holds and let $\Delta > \Delta^*$. There exists a lower-bound price p^* and an upper-bound price p^{**} meeting $h(\bar{t},\bar{t}) - h(q_{\max},\bar{t}) + \Delta \geq p^{**} \geq p^* \geq h(\bar{t},\bar{t}) - h(q_{\min},\bar{t})$ such that for any price $p \in [p^*, p^{**}]$, there exists a sorting equilibrium in BM characterized by a cut-off $\tilde{t}^p$ such that types above this threshold rank school 1 first while types below rank school 2 in first position. Moreover, $\tilde{t}^p > \tilde{t}_{\max}$.*

Corollary 1 *In a (maximum cut-off) sorting equilibrium in BM with a private school, top-type students have a higher probability of accessing the best school than when a private school does not exist. Furthermore, the ex-post quality of the best public school, as well as that of school 2, increases with respect to the case without private school.*

The intuition is the following. Some households assigned by the mechanism to the ghetto school opt out of the public system and send their child to the private alternative, so that the ghetto school loses peer quality. Given that agents who rank the best school first have a higher chance to receive a slot in the ghetto school than those misreporting their preferences, the former strategy becomes less attractive and more agents play the safer one. Formally, the equilibrium cut-off type is higher with the private option, which means the two good public schools have better peer quality and that the best one admits top-types more easily, as compared to the no-private-school case.

Our final result shows the existence of a continuum of equilibria with full income segregation under BM.

Proposition 6 *Suppose A2 holds and that $u(\underline{t}) - u(\underline{t} - p) > \Delta + h(\mathbb{E}t, \underline{t}) - h(\underline{t}, \underline{t})$. For any $\Delta > \Delta^*$ there are two prices $\hat{p}$ and $\hat{p}' > \hat{p}$, such that for any $p \in (\hat{p}, \hat{p}']$ a full segregation equilibrium exists in BM characterized by two thresholds $\hat{t} \in (t_{1/3}, t_{1/2})$ and $t_3 < \hat{t}$ such that all types above (below) $\hat{t}$ rank school 1 (2) first; school 3 is ranked last by every type; (ii) all students assigned to school 3 with type above (below) t_3 opt for the private school, while the rest stay put. These equilibria are sequential.*

Income segregation emerges in this scenario if all households who rank school 1 first have the back up of the private school, that is, if they have enough income to avoid ending up in the ghetto school if rejected from school 1. Put differently, income segregation arises when those who cannot afford the private school misrepresent their preferences to play a safer strategy.[36] The single-crossing condition then holds strictly under A2: because higher income agents have lower marginal utility of income, their utility cost of paying tuition fees p is smaller. Hence their relative valuation of the private school (and so of strategy 1) is larger *even if their kids do not benefit more from school quality than others.*[37]

Remark 4 *Instead of a private school with tuition fees p that induces a threshold t_3, for both preceding propositions we could have considered a "direct" model in which only high types (with t above t_3) had an outside option that is better than the bad school yet worse than a good public school. An example of this could have been the case of a selective private school. Conclusions would have been identical.*

To further illustrate the role of the bad school effect with a private school, we present an example that resembles an equilibrium found by Epple and Romano (1996). In this example, all public schools are equally good ex ante. The private school captures the highest types, yet no public school "elitization" arises.

Example 1 *Let $\Delta = 0$ and set $p = h(q_{\max}, t_{(2/3)}) - h(\mathbb{E}(t \left| t \leq t_{(2/3)} \right.), t_{(2/3)})$. Both under A1 or A2, and both with DA and BM, an equilibrium exists where all types above $t_{(2/3)}$ directly attend the private school; the remaining agents evenly randomize between any possible ordering of the public schools. Ex post we have $\hat{q}_1 = \hat{q}_2 = \hat{q}_3 = \mathbb{E}(t \left| t \leq t_{(2/3)} \right.)$ and $\hat{q}_p = q_{\max}$, and the $t_{(2/3)}$-type household is indifferent between the two strategies.*[38]

[36] Given that school quality differences do not make a difference for the lowest type children, assuming $u(\underline{t}) - u(\underline{t} - p) > \Delta$ simply ensures that their parents are poor enough to prefer school 3 over the private alternative.

[37] Otherwise, the relative valuation of strategies 1 and 2 does not change with type and the single-crossing condition only holds weakly.

[38] A similar example would arise with no private school when school 3 embeds a human capital loss $\Delta = h(q_{\max}, t_{(2/3)}) - h(\mathbb{E}(t \left| t \leq t_{(2/3)} \right.), t_{(2/3)})$. Under A1, both DA and BM sort students in a way that types above $t_{(2/3)}$ attend school 3 and types below attend either school 1 or 2 indistinctly. This gives an interesting reinterpretation of Δ as a cost which allows a public school to become selective.

6 Concluding remarks

This paper has introduced a theory of sorting into public schools with school choice. It is, to the best of our knowledge, the first study on school choice mechanisms that endogenizes preferences and priorities. We showed that the choice of the assignment mechanism, along with the details of the institutional context in which it is applied, are crucial for the resulting distribution of children across public schools and for the degree of equality of opportunities offered by the education system. We thus provided a rigorous theoretical underpinning for the equity concerns expressed by the OECD (2012) and others, even if there are no transport costs or informational asymmetries. Our analysis also offers guidance about how to guarantee equality of opportunities in a context with public sector school choice.

The work presented here invites to natural extension: the consideration of a two-dimensional type space. We explore this richer framework in the appendix. There, richer households have higher willingness to pay than poorer households with the same (ability) type t for the same priority right or for a private school. Cut-off equilibria in our model would become bandwidth equilibria with a monotonic locus partitioning the bidimensional type space. Quality differences across schools would easily emerge as in our main model, with just a mild positive correlation between income and ability. For instance, it could be the case that both a rich low-type household and a poor high-type household send their children to the best public school. Overall, however, a rich high-type household would also enter that school. Thus, the quality of this school would surely remain higher than any other public school quality ex post.

The welfare implications of segregating or mixing students are well known (Arnott and Rowse, 1987; Bénabou, 1996, RES). For that reason, we do not carry out a welfare analysis.

7 Appendix

7.1 Proofs

Proof of Lemma 1. DA: Since it is expected that school 1 will have higher quality than school 2, and given that school 3 is the worst school, all families will submit (weakly dominant strategy) the same ranking of schools: 1,2,3.

The DA algorithm will then assign the slots of school 1 to those students with residence priority in district 1, and the slots of school 2 to the students that have residence in district 2.

BM: Consider the families with residence priority at school 1. For them, ranking school 1 first constitutes part of a weakly dominant strategy, since they have guaranteed acceptance if they put it first in the submitted ranking and this is in their interest because school 1 provides the highest quality of the three. Fixing such a strategy feature for residents in district 1, ranking school 2 in first position constitutes part of a weakly dominant strategy for residents of district 2. Nothing is to be gained by putting school 1 first, since only residents in district 1 will be accepted there. On the contrary, there is a potential loss if residents in district 3 rank school 2 in first position: they take the slots that residents in district 2 miss by applying to school 1 in the first round. Hence, the unique equilibrium (undominated strategies) outcome is the one in which the slots of school 1 are assigned to those students with residence priority in district 1, and the slots of school 2 to the students that have residence in district 2. ■

Proof of Proposition 1. The proof trivially derives from lemma 1. ■

Proof of Proposition 2. DA: Suppose that $\hat{q}_1 > \hat{q}_2 > \hat{q}_3$. Since DA is strategy-proof, all agents would report the true ranking among schools. Consequently, all agents would have equal chances to access any of the schools. This leads to $q_1 = q_2 = q_3$, a contradiction. Suppose $\hat{q}_1 > \hat{q}_2 = \hat{q}_3$. All agents would rank school 1 first and schools 2 and 3 would fill their slots with the rejected students. But then $q_1 \leq \max\{q_2, q_3\}$, a contradiction. Finally, suppose $\hat{q}_1 = \hat{q}_2 > \hat{q}_3$. All agents would rank school 3 last, and that school would fill its capacity with the rejected students from schools 1 and 2. But then $q_3 \geq \min\{q_1, q_2\}$, a contradiction.

BM: Let $j = 3$ be the unique worst school, that is $\hat{q}_3 < \min\{\hat{q}_1, \hat{q}_2\}$. Then no equilibria exists in which a positive mass of households ranks school 3 other than last. Therefore the quality of school 3 will be defined by the students who are rejected via fair lotteries from both schools 1 and 2. Consequently, q_3 must be a weighted average of q_1 and q_2, which contradicts school 3 being the worst one. Similarly, we cannot have $\hat{q}_1 > \hat{q}_2 = \hat{q}_3$. In that case everyone would optimally rank school 1 first and we would have $q_1 \leq \max\{q_2, q_3\}$, again, a contradiction. ■

Proof of Proposition 3(a). The proof makes use of the following lemma:

Lemma A1. *Let WLOG $\hat{q}_1 \geq \hat{q}_2$ and assume $\Delta > \Delta^*$. Then in equilibrium under BM we have $q_1 \geq q_3 \geq q_2$.*

Proof of Lemma A1. Since $\Delta > \Delta^*$, it is clear that school 3 is the worst one for every household type. Therefore, no equilibria exists in which a positive mass of households ranks school 3 other than last. Then the quality of school 3 will be defined by the students who are rejected via fair lotteries from both schools 1 and 2. Consequently, any consistent q_3 must be a weighted average of q_1 and q_2.

. When submitting a ranking in BM, parents can restrict attention to two different ranking strategies. School 3 is the worst school for every parent, and there is no strategic reason to put it in a position other than last. Should that be done, the chances of going to the worst school would be increased, and the chances to go to any other school would be reduced. So the relevant strategy space for parents is simplified to a set with these two elements: "put school 1 first, 2 second and 3 last", or "put school 2 first, 1 second and 3 last", denoted as $s \in \{1, 2\}$ respectively. WLOG we analyze the case where $\hat{q}_1 > \hat{q}_2$. Let m_s denote the mass of parents using strategy s. It is clear that $m_2 = 1 - m_1$. In equilibrium we always have $m_2 < 1/2 < m_1$ because the chances to get access at school 2 must be higher than the chances to be accepted at school 1 (otherwise all parents would put school 1 in first position). When computing the optimal list there are two cases to consider.

Case 1: Both schools 1 and 2 give all their slots in the first round of the assignment procedure ($m_2 \geq 1/3$). In such a case, parents playing strategy s have a probability $1/3m_s$ of having their children accepted at school s, $1 - 1/3m_s$ of having their children assigned to school 3 and zero chance at the remaining school. The expected utility is $x + h(\hat{q}_s, t)/3m_s + (h(\hat{q}_3, t) - \Delta)(1 - 1/3m_s)$ for a t-type parent playing strategy s. A t-type parent chooses strategy 1 if

$$\frac{h(\hat{q}_1, t) - h(\hat{q}_3, t) + \Delta}{h(\hat{q}_2, t) - h(\hat{q}_3, t) + \Delta} > \frac{m_1}{m_2} \tag{1}$$

and she chooses strategy 2 if the inequality is reversed. We apply the following single-crossing lemma:

Lemma A2. *Let $\hat{q}_1 > \hat{q}_2$ and $\Delta > \Delta^*$. Then $\frac{h(\hat{q}_1,t)-h(\hat{q}_3,t)+\Delta}{h(\hat{q}_2,t)-h(\hat{q}_3,t)+\Delta}$ is increasing in t.*

Proof of Lemma A2. Using lemma A1 (consistency of beliefs implies

$\hat{q}_3 \geq \hat{q}_2$), $h_q \geq 0$ and $h_{qt} > 0$, we are done, since we can rearrange

$$\frac{h(\hat{q}_1,t) - h(\hat{q}_3,t) + \Delta}{h(\hat{q}_2,t) - h(\hat{q}_3,t) + \Delta} = \frac{h(\hat{q}_1,t) - h(\hat{q}_3,t) + \Delta}{\Delta - (h(\hat{q}_3,t) - h(\hat{q}_2,t))}$$

Both the numerator and the denominator are positive. The numerator is increasing in t and the denominator is decreasing in t.

By lemma A2, if a t-type parent best-responds with strategy 1 and $t' > t$, a t'-type parent also chooses strategy 1 optimally. Likewise, if a t-type parent chooses strategy 2 and $t" < t$, a $t"$-type parent chooses strategy 2 as well. This suggests an equilibrium characterized by a threshold $\hat{t}$ (with $1/2 > \Phi(\hat{t}) \geq 1/3$) such that types above it play strategy 1 and the types below play strategy 2. This threshold is characterized by

$$\frac{h(q_1(\hat{t}),\hat{t}) - h(q_3(\hat{t}),\hat{t}) + \Delta}{h(q_2(\hat{t}),\hat{t}) - h(q_3(\hat{t}),\hat{t}) + \Delta} = \frac{[1-\Phi(\hat{t})]}{\Phi(\hat{t})}$$

where $q_1(\hat{t}) \equiv \mathbb{E}(t \,|t \geq \hat{t})$, $q_2(\hat{t}) \equiv \mathbb{E}(t \,|t \leq \hat{t})$, and $q_3(\hat{t}) \equiv 3\mathbb{E}t - q_1(\hat{t}) - q_2(\hat{t})$ (since consistency requires $[q_1(\hat{t}) + q_2(\hat{t}) + q_3(\hat{t})]/3 = \mathbb{E}t$). We rearrange this equation as

$$G_1(\hat{t}) \equiv \frac{\Phi(\hat{t})}{1-2\Phi(\hat{t})} h(q_1(\hat{t}),\hat{t}) - \frac{1-\Phi(\hat{t})}{1-2\Phi(\hat{t})} h(q_2(\hat{t}),\hat{t}) + h(q_3(\hat{t}),\hat{t}) = \Delta \qquad (2)$$

Case 2: School 1 gives all of its slots in the first round and school 2 does not ($m_2 < 1/3$). Parents playing strategy 2 have their children accepted at school 2 for sure, obtaining an expected utility $x + h(\hat{q}_2, t)$ depending on their types t. Parents playing strategy 1 have a probability $1/3m_1$ of having their children accepted at school 1, $\frac{1/3-m_2}{m_1} = 1 - \frac{2}{3m_1}$ of having their children assigned to school 2 and a remaining $1/3m_1$ chance of sending their children to school 3. So if the parent's type is t, the expected utility from playing strategy 1 is $x + h(\hat{q}_2, t) + \frac{h(\hat{q}_1,t)+h(\hat{q}_3,t)-\Delta-2h(\hat{q}_2,t)}{3m_1}$. By comparing it it to $x + h(\hat{q}_2, t)$, a t-type parent chooses strategy 1 if

$$h(\hat{q}_1,t) - 2h(\hat{q}_2,t) + h(\hat{q}_3,t) > \Delta$$

and she chooses strategy 2 if the inequality is reversed. Using lemma 3 ($\hat{q}_3 \geq \hat{q}_2$) and $h_{qt} > 0$, the left-hand side is increasing in t. Moreover, notice that in this case we have $q_3 = q_1$ (no types ranking school 2 first are assigned to school 3). This again suggests an equilibrium characterized by a threshold $\hat{t}$ (with $\Phi(\hat{t}) < 1/3$) such that types above it play strategy 1 and the types below play strategy 2. This threshold is characterized by

$$G_2(\hat{t}) \equiv h(q_1(\hat{t}), \hat{t}) - 2h(\tilde{q}_2(\hat{t}), \hat{t}) + h(\tilde{q}_3(\hat{t}), \hat{t}) = \Delta \tag{3}$$

with $q_1(\hat{t}) \equiv \mathbb{E}(t \left| t \geq \hat{t} \right.)$, $\tilde{q}_3(\hat{t}) \equiv q_1(\hat{t})$ and $\tilde{q}_2(\hat{t}) \equiv 3\mathbb{E}t - 2q_1(\hat{t})$ (since $[q_1(\hat{t}) + \tilde{q}_2(\hat{t}) + \tilde{q}_3(\hat{t})]/3 = \mathbb{E}t$).
Compiling both cases, we define the function $G : [\underline{t}, t_{(1/2)}) \rightarrow \mathbb{R}_+$ as

$$G(t) \equiv \begin{cases} G_2(t) \text{ if } \Phi(t) < 1/3 \\ G_1(t) \text{ if } \Phi(t) \geq 1/3 \end{cases}$$

Notice that G is continuous on its domain since $G_1(t) = G_2(t)$ when $\Phi(t) = 1/3$. There would be an equilibrium characterized by a threshold $\tilde{t}$ below the median type if this $\tilde{t}$ satisfies

$$G(\tilde{t}) = \Delta$$

Given the assumptions on h, we can prove that the Boston Mechanism contains an equilibrium with sorting between schools 1 and 2.
Existence: Since $h(\cdot, \underline{t})$ is a constant function, $G(\underline{t}) = 0$. Around the median $t_{(1/2)}$, we have $\lim\limits_{t \rightarrow t_{(1/2)}} G(t) = \infty$. Given the continuity of G, the intermediate value theorem applies to show existence.
Segregation arises from the cut-off equilibrium strategy profile. When $\Phi(\tilde{t}) < 1/3$ sorting is not full because a positive mass of students ranking school 1 first are assigned to school 2. However, there is partial sorting because Φ_2 is a weighted average between $\Phi_1 = \Phi(\cdot \left| t \geq \tilde{t} \right.)$ and $\Phi_2 = \lneqq(\cdot \left| t \leq \tilde{t} \right.)$, hence $\Phi_1 FOSD \Phi_2$.
The equilibrium is **sequential** because a sequence of beliefs $(q_1^n, q_2^n, q_3^n)_{n \in \mathbb{N}} \rightarrow (q_1, q_2, q_3)$ with $q_1^n > q_2^n$ induces a sequence of best response profiles $R^n : D \times \Xi \rightarrow \mathcal{P}(\{s_1, s_2, s_3\})$ that can be characterized by a sequence of thresholds $\tilde{t}^n$ (types above the threshold rank school 1 first, all types below rank school 2 first) such that $\tilde{t}^n \rightarrow \tilde{t}$.
Uniqueness emerges when Δ is high enough because there exists $t'' < t_{(1/2)}$ such that both (1) $G(t)$ is strictly increasing in $t \in [t'', t_{(1/2)})$ (since $\lim\limits_{t \rightarrow t_{(1/2)}} G(t) = \infty$) and (2) for any $\tau < t'' < \tau'$ we have $G(\tau) < G(\tau')$ (again

for the same reason). It is enough to set $\Delta > G(t'')$ to obtain uniqueness of $\tilde{t}$.

Full sorting if Δ is high enough: observing the function G_2, one can set $\Delta > 2[h(q_{\max}, \bar{t}) - h(q_{\min}, \bar{t})]$, thus $G(\tilde{t}) = \Delta$ cannot be satisfied if $\Phi(\tilde{t}) < 1/3$ and sorting must be full. ■ ■

Proof of Proposition 3(b). Since $\Delta > \Delta^*$, all agents rank school 3 third. As in the proof of Proposition 2, suppose that $\hat{q}_1 > \hat{q}_2 > \hat{q}_3$. Since DA is strategy-proof, all agents would report the true ranking among schools. Consequently, all agents would have equal chances to access any of the schools. This leads to $q_1 = q_2 = q_3$, a contradiction. Suppose $\hat{q}_1 > \hat{q}_2 = \hat{q}_3$. All agents would rank school 1 first and schools 2 and 3 would fill their slots with the rejected students. But then $q_1 \leq \max\{q_2, q_3\}$, a contradiction. Finally, suppose $\hat{q}_1 = \hat{q}_2 > \hat{q}_3$. All agents would rank school 3 last, and that school would fill its capacity with the rejected students from schools 1 and 2. But then $q_3 \geq \min\{q_1, q_2\}$, a contradiction.

■

Proof of Lemma 3. The type indifferent between school j and the private alternative, t_j, satisfies:

$$h(q_p, t_j) - h(q_j, t_j) - [u(t_j) - u(t_j - p) - \mathbf{1}_{j=3} \cdot \Delta] = 0. \tag{4}$$

The result follows immediately either if h is supermodular, since $q_p > q_j$, or if u is strictly concave.

■

Proof of Proposition 4. Let $\hat{q}_1 = \hat{q}_2 > \hat{q}_3$. There is a (dominant strategy) equilibrium in which all agents rank school 1 first, school 2 second and school 3 last. DA allocates students across schools randomly so that average peer quality of children admitted to each school is equal to $\mathbb{E}t$, regardless the value of Δ. Then, under assumption A1, $p \leq \Delta + h(\bar{t}, \bar{t}) - h(\mathbb{E}t, \bar{t})$ ensures the existence of a cut-off value of t, $t_3 \leq \bar{t}$, such that types above t_3 assigned to school 3 opt for the private school at the final stage while types below t_3 stay put. Hence $\Phi_3 = \Phi(\cdot | t \leq t_3)$. Next, note that $p \geq h(\bar{t}, \bar{t}) - h(\mathbb{E}t, \bar{t})$ is a sufficient condition for no agent assigned to schools 1 and 2 to purchase private education. In that case, we have ex-post $q_1 = q_2 = \mathbb{E}t$, as in proposition 2, and the distribution of types assigned to both schools 1 and 2, Φ, first-order stochastically dominates Φ_3. If the price of the private school falls sufficiently below that level, then some high types assigned to schools 1 or 2 will opt for the private alternative so that $t_1 = t_2 < \bar{t}$. In that case, since $\Delta > 0$, it is straightforward to show that $t_3 < t_1 = t_2$, so that, again, $\Phi_1 = \Phi_2 = \Phi(\cdot|t \le t_1)$ first order stochastically dominates $\Phi_3 = \Phi(\cdot|t \le t_3)$. For sequentiality, simply let $q_1^n > q_2^n > q_3^n$ all converge to $q_j^n \to \mathbb{E}(t|t \le t_j)$. $\Delta > 0$ forces $t_3 < t_1 = t_2$. For all n the proposed equilibrium strategy profile holds, and the ex-post qualities coincide with those arising from the equilibrium. ■

Proof of Proposition 5. $p^* \ge h(\bar{t}, \bar{t}) - h(q_{\min}, \bar{t})$ ensures that students assigned to schools 1 and 2 do not take the private school option. $p^{**} \le h(\bar{t}, \bar{t}) - h(q_{\max}, \bar{t}) + \Delta$ ensures that $t_3 < \bar{t}$. Let $\hat{t}$ be a candidate equilibrium cut-off. In the base model without a private school, we defined the function $G(\hat{t})$ whose value depends on $\Phi(\hat{t})$ and on the ex-post qualities $q_1(\hat{t})$, $q_2(\hat{t})$ $(\tilde{q}_2(\hat{t}))$ and $q_3(\hat{t})$ $(\tilde{q}_3(\hat{t}))$. Here, we use the same notation just adding superscript p, jointly with $q_p(\hat{t})$, the ex-post quality of the private school, to define the analogous function $G^p(\hat{t})$:

$$G^p(\hat{t}) = \begin{cases} h(q_1^p(\hat{t}), \hat{t}) - 2h(\tilde{q}_2^p(\hat{t}), \hat{t}) + \\ + \max\{h(\tilde{q}_3^p(\hat{t}), \hat{t}) - \Delta, h(q_p(\hat{t}), \hat{t}) - p\} \text{ if } t < t_{(1/3)} \\ \frac{\Phi(\hat{t})}{1-2\Phi(\hat{t})} h(q_1^p(\hat{t}), \hat{t}) - \frac{1-\Phi(\hat{t})}{1-2\Phi(\hat{t})} h(q_2^p(\hat{t}), \hat{t}) + \\ + \max\{h(q_3^p(\hat{t}), \hat{t}) - \Delta, h(q_p(\hat{t}), \hat{t}) - p\} \text{ if } t \ge t_{(1/3)} \end{cases}$$

The proof requires the following lemma:

Lemma A3. Set $p \in [p^*, p^{**}]$. Consider a cut-off $t \in (\underline{t}, t_{(1/2)})$ such that types above it rank school 1 in first position while types below rank school 2 first. There exists $t_3^* \in (t, \bar{t})$ such that for any $t_3 \in (t_3^*, \bar{t})$: 1) the ex-post qualities of schools 1 and 2 remain unaltered regardless the presence of a private school, 2) the ex-post quality of school 3 diminishes with a private school, 3) $\min\{q_1^p(t), q_p(t)\} \ge \tilde{q}_3^p(t) \ge \tilde{q}_2^p(t)$ and $\min\{q_1^p(t), q_p(t)\} \ge q_3^p(t) \ge q_2^p(t)$ (single-crossing conditions apply) and 4) $G^p(t) < G(t) - \Delta$ for all t. ■

Proof of Lemma A3.
Let $t \ge t_{(1/3)}$ and consider any $t_3^* > t$. Then, the ex-post quality of school 1 (respectively, 2) is the same with and without the private school: the expected type conditional on being above (below) t. $q_3(t)$ is a weighted average between $q_1(t)$ and $q_2(t)$, while $q_3^p(t)$ is a weighted average, with the same weights as $q_3(t)$, of the following two elements: a) the expected type conditional on it being in the interval $[t, t_3^*]$, and b) $q_2(t)$. It follows that

$q_3(t) > q_3^p(t) \geq q_2(t) = q_2^p(t)$.
Since naturally $\min\{q_1^p(t), q_p(t)\} > q_3^p(t)$, single crossing arises (analogously to the case with no private school: $\frac{\Phi(\hat{t})}{1-2\Phi(\hat{t})}\ h(q_1^p(\hat{t}), \tau) - \frac{1-\Phi(\hat{t})}{1-2\Phi(\hat{t})}\ h(q_2^p(\hat{t}), \tau)$ $+ \max\{h(q_3^p(\hat{t}), \tau) - \Delta, h(q_p(\hat{t}), \tau) - p\}$ is increasing in τ).
Moreover, since $q_1^p(t) = q_1(t)$, $q_2^p(t) = q_2(t)$ and $q_3(t) > q_3^p(t)$ and given that the t-type household does not choose the private school, it also follows that $G^p(t) < G(t) - \Delta$.
Let $t < t_{(1/3)}$. We show the existence of a $t_3^* \in (t, \bar{t})$ such that all the desired properties are met. $q_1(t)$ (and also $q_1^p(t)$) still equals the expected type conditional on being above t. $\tilde{q}_2(t)$ (as well as $\tilde{q}_2^p(t)$) equals $3\Phi(t)$ times the expected type conditional on being below t plus $1 - 3\Phi(t)$ times $q_1(t)$. While $\tilde{q}_3(t) = q_1(t)$, $\tilde{q}_3^p(t)$ is the expected type conditional on it lying on the interval $[t, t_3^*]$. It follows again that $q_1^p(t) = q_1(t) \geq \tilde{q}_3(t) > \tilde{q}_3^p(t)$. And for t_3^* high enough we have $\tilde{q}_3^p(t) \geq \tilde{q}_2(t) = \tilde{q}_2^p(t)$. Single crossing arises because $q_1^p(t) > \tilde{q}_3^p(t) \geq \tilde{q}_2^p(t)$ (as in the case with no private school: $h(q_1^p(\hat{t}), \tau) - 2h(\tilde{q}_2^p(\hat{t}), \tau) + \max\{h(\tilde{q}_3^p(\hat{t}), \tau) - \Delta, h(q_p(\hat{t}), \tau) - p\}$ is increasing in τ). $q_1^p(t) = q_1(t)$, $\tilde{q}_2^p(t) = \tilde{q}_2(t)$, $\tilde{q}_3(t) > \tilde{q}_3^p(t)$ and t choosing to stay put in school 3 if assigned there implies $G^p(t) < G(t) - \Delta$.
It is immediate that the aforementioned properties would hold with any $t_3 > t_3^*$, since $q_3^p(t)$ (and $\tilde{q}_3^p(t)$) increases with t_3 while never being higher than $q_3(t)$ (or $\tilde{q}_3(t)$).
We can now conclude the proof of the proposition. We just need to ensure that t_3^* is below $\tilde{t}_{\max}$ so that lemma A3 holds. It follows that $G^p(\tilde{t}_{\max}) < G(\tilde{t}_{\max}) - \Delta = 0$. $G^p(\cdot)$ is continuous and $\lim\limits_{t \to t_{(1/2)}} G^p(t) = \infty$, so the intermediate value theorem applies to show existence of at least one cut-off $\tilde{t}^p \in (\tilde{t}_{\max}, t_{(1/2)})$ meeting $G^p(\tilde{t}^p) = 0$. We obtain the same result for each $t_3 \in (t_3^*, \bar{t})$. A different price is chosen for each t_3 so as to make t_3-type households indifferent between school 3 and the private school, giving the required range of prices.

$$\begin{aligned} \text{Either } h(\tilde{q}_3^p(\hat{t}), t_3) - \Delta &= h(q_p(\hat{t}), t_3) - p \\ \text{or } h(q_3^p(\hat{t}), t_3) - \Delta &= h(q_p(\hat{t}), t_3) - p \end{aligned}$$

Obviously $\hat{t}$ depends on t_3, and p depends on both $\hat{t}$ and t_3. Since p varies continuously (although maybe not monotonically) with both $\hat{t}$ and t_3, the desired range of prices constitutes an interval $[p^*, p^{**}]$.

Proof of Proposition 6. We initially assume: (i) that students assigned to schools 1 and 2 stay put, and (ii) that $t_3 \leq \hat{t}$; and then prove that both (i) and (ii) hold in the equilibria we find.[39] Under (i) the quality of schools 1 and 2 is determined at the school assignment stage and we can write the quality of these two schools as a function of the cut-off: $q_j^p(\hat{t})$, $j = 1, 2$. Notice as well that that under (ii) $q_2^p(\hat{t}) < q_p(\hat{t}) \leq q_1^p(\hat{t})$ for any $\hat{t} \in (\underline{t}, t_{1/2})$.

We start with the last stage of the game, when the quality of school 3 as that of the private one are determined. Let $\delta(t, p) = u(t) - u(t - p)$ be the utility cost of paying p units of the numeraire. $\delta(t, p)$ is positive for any $p > 0$, continuous, and satisfies $\delta(t, t) = \infty$ under assumption A2. Equation (5) determines the type indifferent between school 3 and the private alternative t_3 at the final stage:

$$\delta(t_3, p) + h(q_3^p(\hat{t}, t_3), t_3) - h(q_p(\hat{t}, t_3), t_3) - \Delta = 0 \tag{5}$$

Given Δ and the school assignment cut-off $\hat{t}$, this equation has (at least) one solution $t_3 \in \left[\underline{t}, \overline{t}\right]$ for every price $p \in \left(0, \overline{t}\right)$.[40] Because its LHS is differentiable for all $(t_3, \hat{t}, p)$, by the implicit function theorem, (5) defines a continuous and well-defined function $t_3(p, \hat{t})$ from $\left(0, \overline{t}\right) \times \left[t_{1/3}, t_{1/2}\right]$ onto $(\underline{t}, \overline{t})$ that determines the last stage equilibrium cut-off t_3.

Consider now the school assignment stage. For any $\Delta > \Delta^*$ all agents rank school 3 last in the school assignment stage. Therefore, the relevant strategy space has the same two elements as before: "put school 1 first, 2 second and 3 last", or "put school 2 first, 1 second and 3 last". It is then possible to use the function analog to $G^p(\hat{t})$ which, under assumption A2, we can write as follows for $\hat{t} \in \left[t_{1/3}, t_{1/2}\right]$:

$$H^p(t_3, \hat{t}, p) = \left\{ \begin{array}{c} \frac{\Phi(\hat{t})}{1-2\Phi(\hat{t})} h(q_1^p(\hat{t}), \hat{t}) - \frac{1-\Phi(\hat{t})}{1-2\Phi(\hat{t})} h(q_2^p(\hat{t}), \hat{t}) + \\ + \max\{h(q_3^p(t_3, \hat{t}), \hat{t}) - \Delta, h(q_p(\hat{t}), \hat{t}) - \delta(t, p)\}. \end{array} \right.$$

Let

$$r(t_3, \hat{t}) = \left\{ \begin{array}{c} \frac{\Phi(\hat{t})}{1-2\Phi(\hat{t})} h(q_1^p(\hat{t}), \hat{t}) \\ - \frac{1-\Phi(\hat{t})}{1-2\Phi(\hat{t})} h(q_2^p(\hat{t}), \hat{t}) + h(q_p(t_3, \hat{t}), \hat{t}). \end{array} \right.$$

[39] In fact, no equilibrium exists in which (ii) does not hold.

[40] If for some $\hat{t}$ we have that $u(\underline{t}) - u(\underline{t} - p) + h(\underline{t}, \underline{t}) - h(q_p(\hat{t}, \underline{t}), \underline{t}) - \Delta < 0$ then all types would prefer the private school and $t_3 = \underline{t}$; if instead $u(\overline{t}) - u(\overline{t} - p) + h(q_3(\hat{t}, \overline{t}), \overline{t}) - h(\overline{t}, \overline{t}) - \Delta > 0$ then all types would prefer school 3 to the private option so that $t_3 = \overline{t}$.

Then, provided (ii) holds, $H^p(t_3, \hat{t}, p) = r(t_3, \hat{t}) - \delta\left(\hat{t}, p\right)$ so that a first-stage equilibrium candidate cut-off $\hat{t}$ must satisfy

$$r(t_3, \hat{t}) - \delta(\hat{t}, p) = 0. \tag{6}$$

Plug $t_3(p, \hat{t})$ into (6). While $r(t_3(p, \hat{t}), \hat{t})$ may not be monotone in $\hat{t}$, it is strictly positive for all $\hat{t} \in (t_{1/3}, t_{1/2})$, since $q_2^p(\hat{t}) < q_p(\hat{t}) \leq q_1^p(\hat{t})$ for any $\hat{t} \in (t_{(1/3)}, t_{(1/2)})$. Furthermore, $\lim_{\hat{t} \to t_{1/2}} r(t_3(p, \hat{t}), \hat{t}) = \infty$, since $1 - 2\Phi\left(t_{(1/2)}\right) = 0$. Then, for any $\hat{t} \in (t_{1/3}, t_{1/2})$, there exists a unique price such that (6) holds, because the RHS is positive and continuous and the LHS varies continuously from 0 to ∞ for $p \in (0, \hat{t})$. Again, since its LHS is continuously differentiable for all $(\hat{t}, p)$, by the implicit function theorem, (6) defines a continuously differentiable function $p(\hat{t})$ from $\left(t_{(1/3)}, t_{(1/2)}\right)$ onto $(0, \bar{t})$ such that (6) holds. The set of points $\left(\hat{t}, p(\hat{t}), t_3(p(\hat{t}), \hat{t})\right)$ for $\hat{t} \in (t_{(1/3)}, t_{(1/2)})$ is the set of equilibrium candidates. Henceforth, we simplify notation and use $q_3^p(\hat{t})$ and $q_p(\hat{t})$. We next prove that satisfaction of the relevant single-crossing condition requires $t_3 \leq \hat{t}$, as assumed initially in the proof, and use the result to restrict the set of equilibrium candidates. Take a candidate cut-off $\hat{t}$, fix school qualities accordingly at $q_j^p(\hat{t})$, $j = 1, 2, 3, p$, and calculate the direct derivative of $H^p(t_3, t, p)$ with respect to t. Under assumption A2, type and school quality are independent in the production of human capital and so this derivative is:

$$H^{p\prime}(t, p) = \left\{ \begin{array}{r} 0; \text{ if } t < t_3 \\ u'(t - p) - u'(t) > 0; \text{ otherwise.} \end{array} \right. \tag{7}$$

(7) implies that the relative valuation of strategies 1 and 2 for a given cut-off does not change with type for $t < t_3$, i.e. for types that prefer school 3 over the private school, but rises with income for the rest, i.e. for $t \geq t_3$. Note that the single-crossing condition holds strictly provided $H^{p\prime}(t, p)$ increases with type for $t \geq \hat{t}$, that is, for types at least as high as the cut-off (otherwise, it holds weakly). Hence, given a candidate cut-off $\hat{t}$, a necessary and sufficient condition for preferences over the two strategies to satisfy the single-crossing condition is that t_3 be no larger than $\hat{t}$, which requires:

$$\delta(\hat{t}, p\left(\hat{t}\right)) + h(q_3^p(t_3(\hat{t}), \hat{t}) - h(q_p(\hat{t}), \hat{t}) \leq \Delta \tag{8}$$

For a candidate equilibrium $\hat{t} > t_{1/3}$, the equilibrium condition is

$$\frac{\Phi(\hat{t})}{1 - 2\Phi(\hat{t})} h(q_1^p(\hat{t}), \hat{t}) - \frac{1 - \Phi(\hat{t})}{1 - 2\Phi(\hat{t})} h(q_2^p(\hat{t}), \hat{t}) + h(q_p(\hat{t}), \hat{t}) = \delta(\hat{t}, p\left(\hat{t}\right)). \tag{9}$$

Plugging the LHS of (9) into (8) we obtain that a candidate cut-off $\hat{t} > t_{1/3}$ satisfies the single-crossing condition provided:

$$\frac{\Phi(\hat{t})}{1-2\Phi(\hat{t})}h(q_1^p(\hat{t}),\hat{t}) - \frac{1-\Phi(\hat{t})}{1-2\Phi(\hat{t})}h(q_2^p(\hat{t}),\hat{t}) + h(q_3^p(\hat{t}),\hat{t}) \leq \Delta \tag{10}$$

Let the LHS of (10) be denoted with $s(\hat{t})$ and take its side limit when $\hat{t}$ approaches $t_{(1/3)}$ from greater values:

$$\begin{aligned}\lim_{\hat{t}\to t^+_{(1/3)}} s(\hat{t}) &= \lim_{\hat{t}\to t^+_{(1/3)}} \left[h(q_1^p(\hat{t}),\hat{t}) - 2h(q_2^p(\hat{t}),\hat{t}) + h(q_3^p(\hat{t}),\hat{t})\right] < \\ &< \lim_{\hat{t}\to t^+_{(1/3)}} \left[h(q_1^p(\hat{t}),\hat{t}) - h(q_2^p(\hat{t}),\hat{t})\right]\end{aligned}$$

if $\lim_{\hat{t}\to t^+_{(1/3)}} \left[q_3^p(\hat{t}) - q_2^p(\hat{t})\right] \leq 0$. Moreover,

$$\begin{aligned}&\lim_{\hat{t}\to t^+_{(1/3)}} \left[h(q_1^p(\hat{t}),\hat{t}) - h(q_2^p(\hat{t}),\hat{t})\right] \\ = \;& h(\mathbb{E}(\,t|\,t > t_{(1/3)}), t_{(1/3)}) - h(\mathbb{E}(\,t|\,t < t_{(1/3)}), t_{(1/3)}) < \\ < \;& h(q_{\max}, t_{(1/3)}) - h(q_{\min}, t_{(1/3)}) = \Delta^* \leq \Delta.\end{aligned}$$

We next go back to the last stage to prove existence of an equilibrium with $t_3 < \hat{t}$, denote with $V_j(t_3,\hat{t},p,t)$ $(j = 3, p.)$ the utility derived from school 3 and the private school by a household of type t at $\left(t_3,\hat{t},p\right)$. The previous inequality implies that $\lim_{\hat{t}\to t^+_{(1/3)}} \left(V_p(\hat{t},\hat{t},p,\hat{t}) - V_3(\hat{t},\hat{t},p,\hat{t})\right) > 0$ for any $\Delta \geq \Delta^*$, since in that case $\lim_{\hat{t}\to t^+_{(1/3)}} q_3^p(\hat{t}) = \lim_{\hat{t}\to t^+_{(1/3)}} q_2^p(\hat{t})$. Moreover, the assumption that the lowest type is poor enough to satisfy $u(\underline{t})-u(\underline{t}-p) > \Delta + h(\mathbb{E}t,\underline{t}) - h(\underline{t},\underline{t})$ ensures that $\lim_{\hat{t}\to t^+_{(1/3)}} \left(V_p(\underline{t},\hat{t},p,\underline{t}) - V_3(\underline{t},\hat{t},p,\underline{t})\right) < 0$. Hence, by continuity and the intermediate value theorem, a solution of the last stage problem exists with $t_3 < \hat{t}$ and the single-crossing condition holds strictly in the limit when $\hat{t} \to t^+_{(1/3)}$ for any $\Delta \geq \Delta^*$. By continuity, it also holds for some non-degenerate interval $\left(t_{(1/3)}, \check{t}\,(\Delta)\right)$ where $s(\check{t}\,(\Delta)) = \Delta$. Note that $\check{t}\,(\Delta) < t_{(1/2)}$ for $\lim_{\hat{t}\to t_{(1/2)}} s(\hat{t}) = \infty$.

We continue by proving that all students assigned to schools 1 and 2 stay put, according to the initial assumption in the proof. Agents assigned to school 1 clearly prefer to stay put for any $p > 0$, since $q_p(\hat{t}) \leq q_1^p\left(\hat{t}\right)$ provided $t_3 \leq \hat{t}$. Agents assigned to school 2 have type $t \leq \hat{t}$, since the school is over-subscribed in the first round when $\hat{t} > t_{1/3}$; then, by lemma 3 they all prefer it over the private alternative if:

$$u(\hat{t}) - u(\hat{t} - p) > h(q_p(\hat{t}), \hat{t}) - h(q_2^p(\hat{t}), \hat{t})$$

which, when $\hat{t} \geq t_{1/3}$ and using (9), can be rewritten as

$$\frac{\Phi(\hat{t})}{1 - 2\Phi(\hat{t})} h(q_1^p(\hat{t}), \hat{t}) - \frac{1 - \Phi(\hat{t})}{1 - 2\Phi(\hat{t})} h(q_2^p(\hat{t}), \hat{t}) + h(q_p(\hat{t}), \hat{t}) > h(q_p(\hat{t}), \hat{t}) - h(q_2^p(\hat{t}), \hat{t}).$$

This inequality simplifies to $\frac{\Phi(\hat{t})}{1-2\Phi(\hat{t})}\left[h(q_1^p(\hat{t}), \hat{t}) - h(q_2^p(\hat{t}), \hat{t})\right] > 0$, and so holds for any $\hat{t}$.

We can now establish the set of private school prices such that an equilibrium exists for some $t \in (t_{(1/3)}, \check{t}\,(\Delta)]$. Since $p(t)$ is a continuous function, the extreme value theorem implies it has a maximum and a minimum in $\left[t_{(1/3)}, \check{t}\,(\Delta)\right]$. If the minimum is reached at $t_{(1/3)}$ then there is a half-closed non-degenerate interval of prices with open lower bound $p(t_{(1/3)})$ such that an equilibrium exists, i.e. for which (9) and (10) hold with $\hat{t} \in (t_{(1/3)}, \check{t}\,(\Delta)]$. Otherwise the interval is closed and has a lower bound smaller than $p(t_{(1/3)})$. If the maximum is reached at $\check{t}\,(\Delta)$ then the upper bound price is $p\,(\Delta) = p\left(\check{t}\,(\Delta)\right)$.

These equilibria are sequential because one can construct a sequence of beliefs $(q_1^n, q_2^n, q_3^n, q_p^n)_{n\in\mathbb{N}} \rightarrow \left(q_1^p\left(\hat{t}\right), q_2^p\left(\hat{t}\right), q_3^p\left(\hat{t}\right), q_p\left(\hat{t}\right)\right)$ with $q_1^n \geq q_p^n > q_2^n \geq q_3^n$ which induces a sequence of best response profiles $R^n : D \times \Xi \rightarrow \mathcal{P}(\{s_1, s_2, s_3\})$ and last stage choices that can be characterized by a sequence of pairs of thresholds $\left(\hat{t}^n, t_3^n\right)$ such that $\left(\hat{t}^n, t_3^n\right) \rightarrow \left(\hat{t}, t_3\right)$.

■

7.2 An alternative ghetto effect

We consider a different modeling of the human capital loss produced by being assigned to a bad school. In the main model, the human capital loss is constant over all types. Here, we consider a reduction in quality: if the mean type across the students assigned to school 3 is q_3, then the school quality is δq_3, where $\delta \in (0,1)$ is a "degradation" factor that applies only to school 3. So the bad school effect affects school quality equally for every student.

We assume that $\underline{t} > 0$ and that $h(\cdot, t)$ is defined on $(0, \bar{t}]$ for every t, with $\lim_{q \to 0} h(q,t) = -\infty$ and $h(\underline{t}, t) \geq 0$. Going to a sufficiently degraded bad school produces an enormous human capital loss. Under these assumptions, one can see that the single-crossing conditions ensuring that the Boston Mechanism without priorities generates an equilibrium with sorting hold here, if the degradation factor is low enough.

Lemma 4 *If δ is sufficiently small and $q_1 > q_2$, then both $\frac{h(q_1,t)-h(\delta q_3,t)}{h(q_2,t)-h(\delta q_3,t)}$ and $\frac{h(q_1,t)+h(\delta q_3,t)}{h(q_2,t)}$ are increasing in $t \in [\underline{t}, \bar{t}]$.*

The proof arises immediately because $h(\delta q_3, t)$ becomes negative and sufficiently low. The first ratio is qualitatively equivalent to $\frac{h(q_1,t)-h(q_3,t)+\Delta}{h(q_2,t)-h(q_3,t)+\Delta}$ in the main model, case 1. The second ratio is qualitatively equivalent to $\frac{h(q_1,t)-h(q_3,t)+\Delta}{h(q_2,t)}$, case 2. Then, for δ sufficiently close to 0, we can ensure the existence of a sequential equilibrium with sorting as in the main model. Moreover, for δ sufficiently close to 0, this equilibrium would entail full sorting (just as in the main model with Δ high enough).

7.3 The base model with more than 3 schools

As we have mentioned in section 4, DA with no residence priority nor private schools does not generate sorting between schools regardless the number of schools. It is not clear if the results we found for BM in the same section extend to scenarios with more than three schools. Suppose that we have $J > 3$ equally sized schools and that school J is bad (being assigned there entails a utility loss equal to $\Delta > 0$). Is there an equilibrium with (full) sorting?

Proposition 7 *In BM with no priorities nor private schools, if Δ is sufficiently large, there only exist sequential equilibria with full sorting between every pair of good schools i, j ($i < j$).*

Proof. The first step is to show that with Δ sufficiently large, all schools apart from the bad one give all their slots in the first round in equilibrium. Suppose not. If a strict subset $S \subset \{1, ..., J-1\}$ of good schools do not give all its slots in the first round, then any student who ranks a school from S in first position avoids the punishment Δ for sure. Conditional on ranking any school in the complement of S first, the probability of being assigned to the worst school (hence suffering the utility loss Δ) is on average at least $\frac{1}{J-\#S}$. Setting $\Delta > (J-\#S)[h(\mathbb{E}(t|\Phi(t) \geq (J-1)/J), \bar{t}) - h(\mathbb{E}(t|\Phi(t) \leq 1/J), \bar{t})]$, some types who were not ranking a school from S in first position would be strictly better-off ex ante by doing so. Hence we did not have a best-response profile, a contradiction.

The second step is to show how, when the second round assigns slots only to the worst school, the equilibrium strategy profile R with $q_1 > q_2 > ... > q_{J-1}$ is characterized by cutoffs. In that context, strategies can be simplified to "what good school to rank first": a total of $J-1$ relevant strategies. Let π_i and π_j denote the probabilities of being accepted at schools i and j respectively, conditional on ranking respectively i or j first. Let also $q_i > q_j$. Conditional on ranking i first, the expected payoff for a t-type household is $\pi_i h(q_i, t) + (1-\pi_i)[h(q_J, t) - \Delta]$. An analogous expected payoff form arises when ranking j first. A t-type household prefers to rank i first over ranking j first if $\frac{h(q_i,t)-h(q_J,t)+\Delta}{h(q_j,t)-h(q_J,t)+\Delta} > \frac{\pi_j}{\pi_i}$. The left-hand side ratio is increasing in t if Δ is high enough: the first derivative is positive when

$$\Delta > h(q_J, t) - h(q_j, t) - [h_t(q_J, t) - h_t(q_j, t)]\frac{h(q_i, t) - h(q_j, t)}{h_t(q_i, t) - h_t(q_j, t)}$$

where the right-hand side is upper-bounded even if $q_j \to q_i$ since $\lim\limits_{q_j \to q_i} \frac{h(q_i,t)-h(q_j,t)}{h_t(q_i,t)-h_t(q_j,t)} = \frac{h_q(q_i,t)}{h_{qt}(q_i,t)}$, bounded above as the domain of h is compact and h is doubly differentiable with $h_{qt} \neq 0$. Since $\frac{h(q_i,t)-h(q_J,t)+\Delta}{h(q_j,t)-h(q_J,t)+\Delta}$ is increasing, there exists a threshold $\hat{t}_{ij}$ such that types above it prefer to rank i first over ranking j first while types below prefer the opposite. Moreover, for any triple of good schools i, j, k such that $q_i > q_j > q_k$, we have $\hat{t}_{ij} > \hat{t}_{jk}$ (otherwise no student would rank school j first, contradicting the fact that every good school gives all of its slots in the first round). Therefore we have proven that a series of thresholds $\bar{t} \equiv \hat{t}_{01} > \hat{t}_{12} > \hat{t}_{23} > ... > \hat{t}_{J-2,J-1} > \hat{t}_{J-1,J} \equiv \underline{t}$ (where types in $(\hat{t}_{j,j+1}, \hat{t}_{j-1,j})$ rank school j first) characterize a best-response profile that produces $q_1 > q_2 > ... > q_{J-1}$ ex post.

Existence: The suggested equilibrium would satisfy, for every $j = 1, .., J-2$, and using the notation $\mathcal{T} \equiv (\hat{t}_{01}, \hat{t}_{12}, \hat{t}_{23}, ..., \hat{t}_{J-2,J-1}, \hat{t}_{J-1,J})$:

$$\begin{aligned}
& G_{j,j+1}(\hat{t}_{j,j+1}; \mathcal{T}_{-_{j,j+1}}) \\
\equiv \; & \frac{\Phi(\hat{t}_{j,j+1}) - \Phi(\hat{t}_{j+1,j+2})}{\Phi(\hat{t}_{j-1,j}) - 2\Phi(\hat{t}_{j,j+1}) + \Phi(\hat{t}_{j+1,j+2})} h(q_j(\hat{t}_{j,j+1}; \hat{t}_{j-1,j}), \hat{t}_{j,j+1}) \\
& - \frac{\Phi(\hat{t}_{j-1,j}) - \Phi(\hat{t}_{j,j+1})}{\Phi(\hat{t}_{j-1,j}) - 2\Phi(\hat{t}_{j,j+1}) + \Phi(\hat{t}_{j+1,j+2})} h(q_{j+1}(\hat{t}_{j,j+1}; \hat{t}_{j+1,j+2}), \hat{t}_{j,j+1}) \\
& + h(q_J(\mathcal{T}), \hat{t}_{j,j+1}) \\
= \; & \Delta
\end{aligned}$$

where $q_j(\hat{t}_{j,j+1}; \hat{t}_{j-1,j}) \equiv \mathbb{E}(t | \hat{t}_{j,j+1} \leq t \leq \hat{t}_{j-1,j}) > q_{j+1}(\hat{t}_{j,j+1}; \hat{t}_{j+1,j+2}) \equiv \mathbb{E}(t | \hat{t}_{j+1,j+2} \leq t \leq \hat{t}_{j,j+1})$, and $q_J(\mathcal{T}) \equiv J \cdot \mathbb{E}t - \sum_{j=1,..,J-1} q_j(\hat{t}_{j,j+1};\ \hat{t}_{j-1,j})$ (since $\sum_{j=1}^{J} q_j / J = \mathbb{E}t$).

Such an equilibrium exists because (1) $\lim\limits_{\Phi(t) - \Phi(\hat{t}_{j+1,j+2}) \to 1/J} G_{j,j+1}(t; \mathcal{T}_{-_{j,j+1}}) < \Delta$ for Δ high enough and (2) $\lim\limits_{\Phi(t) \to \frac{\Phi(\hat{t}_{j-1,j}) + \Phi(\hat{t}_{j+1,j+2})}{2}} G_{j,j+1}(t; \mathcal{T}_{-_{j,j+1}}) = \infty$. Since each function $G_{j,j+1}(t; \mathcal{T}_{-_{j,j+1}})$ is continuous in t we can make use of the intermediate value theorem to show existence.

Finally, we show that an equilibrium with $q_i = q_j$ $(i \neq j)$ is not sequential. Such an equilibrium belief cannot be confirmed ex post through a threshold-like strategy profile (where only types above some threshold $\hat{t}_{ij}$ can rank one of the schools first and only types below can rank the other school first.) Yet for any sequence $(q_i^n, q_j^n) \to (q_i, q_j)$ with $q_i^n \neq q_j^n$ we have that the best-response profile R^n is characterized by a threshold $\hat{t}_{ij}^n$ such that only types above the threshold can rank one of the schools first and only types below can rank the other school first. Then the limit of the sequence R^n cannot converge to the initial equilibrium strategies supporting $q_i = q_j$. ■

An important corollary here is that *any sequential equilibrium produces full sorting between any two schools that give all slots in the first round.* Could there be a sequential equilibrium with no sorting when Δ is not sufficiently high? The answer is yes, although not generically. We illustrate the case with one example.

Example 2 *Suppose t is uniformly distributed in the unit interval $[0, 1]$ and set $J = 4$, where school 4 is the bad school. We look for a sequential equilibrium in BM in which in the first round there is a threshold $\hat{t}_{12}$ (types above* *put school 1 in first position, types below rank school 2 first) and in the second round there is a threshold $\hat{t}_{23}$ (types above put school 2 in second position, types below rank school 3 in second position). School 1 is overdemanded in the first round and schools 2 and 3 are overdemanded in the second round. This equilibrium is intended to satisfy $q_1 > q_2 = q_3$ ex post. It is sequential because one can construct a sequence of beliefs $(q_1^n, q_2^n, q_3^n) \to (q_1, q_2, q_3)$ such that $q_2^n > q_3^n$ and $q_1^n/q_j^n > q_1/q_j$, $j \in \{2,3\}$, which induce threshold strategies characterized by cutoffs $(\hat{t}_{12}^n, \hat{t}_{23}^n)$ [41] that converge to $(\hat{t}_{12}, \hat{t}_{23})$ as n grows large. WLOG any other ranking strategy profile yielding $q_2 = q_3$ ex post would not be part of a sequential equilibrium since for that we need cut-off strategy profiles.*

When Φ is the uniform between 0 and 1, we can readily calculate those cutoffs $(\hat{t}_{12}, \hat{t}_{23})$ regardless the shape of h. The first condition is $q_2 = q_3$, that is,

$$\frac{1}{1/4}\left[\hat{t}_{12}\cdot\frac{\hat{t}_{12}}{2} + (1/4 - \hat{t}_{12})\frac{1+\hat{t}_{23}}{2}\right] = \frac{\hat{t}_{12}+\hat{t}_{23}}{2}$$

while the second condition is that applicants for school 2 in the second round and applicants for school 3 in the second round face the same chances of being accepted, namely

$$\frac{1/4 - \hat{t}_{12}}{(3/4 - \hat{t}_{12})\frac{1-\hat{t}_{23}}{1-\hat{t}_{12}}} = \frac{1/4}{(3/4 - \hat{t}_{12})\frac{\hat{t}_{23}-\hat{t}_{12}}{1-\hat{t}_{12}}}$$

The solution to the system of equations is $\hat{t}_{12} \simeq 0.13564$ and $\hat{t}_{23} \simeq 0.72875$, giving ex-post qualities $q_2 = q_3 \simeq 0.4322$ and $q_1 = q_4 \simeq 0.5678$.

Remarkably, both $\hat{t}_{12}$ and $\hat{t}_{23}$ do not depend on the function h. This makes this kind of equilibria non-generic, since there is a last condition that has to be met. The condition is that a $\hat{t}_{12}-$type household must be indifferent between ranking school 2 first and ranking school 1 first, i.e.

$$2h(q_1, \hat{t}_{12}) - 2h(q_2, \hat{t}_{12}) = \Delta$$

For instance, if we postulate a Cobb-Douglass human capital function $h(q,t) = q^\alpha t^\beta$ with parameters $\alpha, \beta > 0$ and we denote the family of all parameters (α, β, Δ) with Π, and we endow this set with any nonatomic measure

[41] For this we assume that $\frac{h(q_2^n,t)-h(q_4,t)+\Delta}{h(q_3^n,t)-h(q_4,t)+\Delta}$ is increasing in t. It is enough to set Δ big enough or to postulate a specific shape of h, for instance a Cobb-Douglass function.

λ *such that* $\lambda(\Pi) = 1$, *we would obtain* $\lambda(\{(\alpha, \beta, \Delta) \in \Pi : 2q_1^\alpha \hat{t}_{12}^\beta - 2q_2^\alpha \hat{t}_{12}^\beta = \Delta\}) = 0$.

Conjecture 3 *Generically, for any fixed distribution of types* Φ *we do not have a sequential equilibrium with no sorting between some pair of good schools.*

7.4 Two-dimensional characteristics space

Under assumption A2, the willingness to pay for residence and schooling in richer households is , ceteris paribus, higher than the willingness to pay in poorer households. In that case, a two-dimensional type space is useful, since it not only considers ability (our "type" t) but also wealth (denoted by y). This subsection extends some results in considering a model with two income levels, H and L, where $H > L$. Conditional on the income level y, the ability distribution is $\Phi(t|y)$. We assume that there is positive correlation between income and ability in such a way that $\Phi(\cdot|H)$ $FOSD$ $\Phi(\cdot|L)$. A mass $\lambda \in (0, 1/2)$ of households has income H and the rest have income L. In order to talk about sorting of abilities across schools, we analyze each subpopulation (high- and low-income households separately). The definitions in the paper can be used for each subpopulation. $\Phi_j(t|y)$ would denote the distribution of ability types among those attending school j conditional on having income y. Accordingly, the ex-post school quality is $\hat{q}_j = \lambda \mathbb{E}_{\Phi_j(\cdot|H)} t + (1 - \lambda) \mathbb{E}_{\Phi_j(\cdot|L)} t$.

Residence priorities

Regardless the mechanism we choose, the only prediction of the model is that of full sorting for each subpopulation. There are cutoffs $a_H \leq b_H$ for the $H-$income subpopulation and $a_L \leq b_L$ for the $L-$income subpopulation such that ability types below a_y choose to reside in district 3 (and they attend school 3), types between a_y and b_y choose to live in district 2 (and they attend school 2), and types above b_y choose district 1 for residence (and they attend school 1). Cutoffs are chosen so that $\hat{q}_1 > \hat{q}_2 > \hat{q}_3$, and rents $r_1 > r_2 > r_3 = 0$ serve to clear the residential market. Moreover, we have $b_H < b_L$, since concavity of u implies $u(H - r_2) - u(H - r_1) < u(L - r_2) - u(L - r_1)$ (regardless the ability type, richer families have lower payoff loss when spending more than poorer families do). For the same reason we have $a_H < a_L$. In extreme cases we could have $a_L = \bar{t}$ or $b_H = \underline{t}$. If school 3 is considered idiosyncratically bad (with direct utility loss Δ), rents r_1 and r_2

would become higher, although not necessarily by an exact amount Δ as in the baseline model.

Private school and no priorities

The variable of interest is the ability type that is indifferent between attending the bad school (school 3) and paying for attending the private school, denoted as t_3 in the baseline model. Here, this cutoff differs across income types, obtaining $t_3^H < t_3^L$ (again, regardless the ability type, richer families have more willingness to pay than poorer families). With t_3^H high enough (with a sufficiently expensive private school or sufficiently low H), single crossing conditions hold for both income types, so that ability cut-off types t_y characterize best responses. That is, $y-$income households rank school 1 first if the ability type lies above t_y, else they rank school 2 in first position. Accordingly, for sufficiently expensive private schools, there exists an equilibrium cut-off type $\tilde{t} < t_3^H < t_3^L$ that does not differ across income types ($\tilde{t} = \tilde{t}_H = \tilde{t}_L$). Being richer only buys families a way to avoid the bad school, but it does not interfere with the assignment in good public schools.

A more interesting case arises when the private school is overly expensive for poor families but affordable for richer families. In an extreme illustrative case we could assume $t_3^H = \underline{t}$. This could be done by properly increasing H so that $u(H-p) \geq u(H) - \Delta$ (recall that a $\underline{t}-$type household does not care about school quality differences). But then, all rich households face less risk than poorer households since not being admitted in a good public school has as a consequence being enrolled in the private school, as compared to the bad school. Consequently, rich households would tend to bet for school 1 rather than the safer option of school 2. In equilibrium we would have $\tilde{t}_H < \tilde{t}_L < t_3^L$. The baseline model predicted an "ability elitization" of school 1 (top ability types get more chances at school 1), as compared to a scenario with no private school. When we introduce income differences and non quasilinear utilities, there is also an "income elitization" effect.

Proposition 8 *Fix L and let $\Delta \geq h(\bar{t}, \bar{t}) - h(q_{\min}, \bar{t})]$. If H is high enough, there exists a lower-bound price p^* and an upper-bound price $p^{**} \geq p^*$ such that for any price $p \in [p^*, p^{**}]$ there exists an equilibrium in BM characterized by cutoffs $\tilde{t}_H^p < \tilde{t}_L^p$ such that households with income $y \in \{L, H\}$ rank school 1 first if their ability types are above $\tilde{t}_y^p$, and they rank school 2 in first position otherwise. ($\tilde{t}_{\max}^p$ denotes the maximum equilibrium cut-off in the game with a p-priced private school in a scenario where H is set equal to L).*

Proof. For p high enough we have t_3^L sufficiently high so that we make sure that the cut-off type $\tilde{t}_L^p$ with income L does not choose the private school against school 3 ($u(L-p) << u(L) - \Delta$). Setting H high enough, we make sure that $u(H-p) \geq u(H) - \Delta$ and then $t_3^H = \underline{t}$. In both income types, it can be checked that single crossing conditions apply: if an ability type chooses to rank school 1 first, so does a higher ability type; if an ability type chooses to rank school 2 first, so does a lower ability type. This allows us to search for income-dependent cut-off types $\tilde{t}_H^p$ and $\tilde{t}_L^p$ meeting $G_H^p(\tilde{t}_H^p, \tilde{t}_L^p) = G_L^p(\tilde{t}_H^p, \tilde{t}_L^p) = 0$ where ■

$$G_y^p(\hat{t}_H, \hat{t}_L) = \begin{cases} h(q_1^p(\hat{t}_H, \hat{t}_L), \hat{t}_y) - 2h(q_2^p(\hat{t}_H, \hat{t}_L), \hat{t}_y) + \\ + \max\{h(q_3^p(\hat{t}_H, \hat{t}_L), \hat{t}_y) - \Delta, \\ h(q_p(\hat{t}_H, \hat{t}_L), \hat{t}_y) + u(y-p) - u(y)\} \\ \text{if } \lambda\Phi(\hat{t}_H|H) + (1-\lambda)\Phi(\hat{t}_L|L) < 1/3 \\ \frac{\lambda\Phi(\hat{t}_H|H)+(1-\lambda)\Phi(\hat{t}_L|L)}{1-2[\lambda\Phi(\hat{t}_H|H)+(1-\lambda)\Phi(\hat{t}_L|L)]} h(q_1^p(\hat{t}_H, \hat{t}_L), \hat{t}_y) - \\ \frac{1-[\lambda\Phi(\hat{t}_H|H)+(1-\lambda)\Phi(\hat{t}_L|L)]}{1-2[\lambda\Phi(\hat{t}_H|H)+(1-\lambda)\Phi(\hat{t}_L|L)]} h(q_2^p(\hat{t}_H, \hat{t}_L), \hat{t}_y) + \\ + \max\{h(q_3^p(\hat{t}_H, \hat{t}_L), \hat{t}_y) - \Delta, \\ h(q_p(\hat{t}_H, \hat{t}_L), \hat{t}_y) + u(y-p) - u(y)\} \\ \text{if } \lambda\Phi(\hat{t}_H|H) + (1-\lambda)\Phi(\hat{t}_L|L) \geq 1/3 \end{cases}$$

$q_j^p(\hat{t}_H, \hat{t}_H)$'s are the school qualities when the price for the private school is p if the cut-off candidates are $\hat{t}_H$ and $\hat{t}_H$. $q_p(\hat{t}_H, \hat{t}_L)$ is the quality of the private school under price p with these cutoffs.

In case there exists a cut-off equilibrium, it cannot be the case that $\tilde{t}_H^p \geq \tilde{t}_L^p$ since we would have $G_H^p(\tilde{t}_H^p, \tilde{t}_L^p) > G_L^p(\tilde{t}_H^p, \tilde{t}_L^p)$. We then show that an equilibrium with $\tilde{t}_H^p < \tilde{t}_L^p$ exists. On the one hand we have that $G_H^p(\underline{t}, \cdot) < 0$ if $(1-\lambda)\Phi\left(\hat{t}_L|L\right) < 1/2$, and $G_L^p(\cdot, \underline{t}) < 0$ for any $\hat{t}_H$. Also, notice that if $\lambda\Phi(\hat{t}_H|H) + (1-\lambda)\Phi(\hat{t}_L|L) = 1/2$ and $\hat{t}_H > \underline{t}$ then $G_H^p(\hat{t}_H, \hat{t}_L) = G_L^p(\hat{t}_H, \hat{t}_L) = \infty$. Select one such pair $(\hat{t}_H, \hat{t}_L)$ with $\lambda\Phi(\hat{t}_H|H) + (1-\lambda)\Phi(\hat{t}_L|L) = 1/2$. Continuity of G's almost everywhere and the intermediate value theorem imply that in the segment with extremes $(\underline{t}, \underline{t})$ and $(\hat{t}_H, \hat{t}_L)$ there are two points $(t_H, f(t_H))$ and $(g(t_L), t_L)$ such that $G_L^p(t_H, f(t_H)) = G_H^p(g(t_L), t_L) = 0$. This defines two functions f and g which can be picked to be continuous almost everywhere due to the continuity of G's almost everywhere. We show that f and g intersect at some point $(\tilde{t}_H^p, \tilde{t}_L^p)$, a cut-off equilibrium. There is only one discontinuity of G_H^p around $(\underline{t}, t_L^m)$ where $(1-\lambda)\Phi(t_L^m|L) = 1/2$, thus $\lim_{t_L \to t_L^m} g(t_L) = \underline{t}$. Notice that $f(\underline{t}) < t_L^m$ since $G_L^p(\underline{t}, t_L^m) = \infty$. So when

$\hat{t}_H \to \underline{t}$ (hence we approach flat line from the origin $(\underline{t}, \underline{t})$), g lies at the right from f. If we go to the 45 degree line, it is easy to observe that g lies at the left from f on that line, since $G_H^p(t,t) > G_L^p(t,t)\ \forall t < t_{(1/2)}$. Continuity of f and g everywhere except for $(\underline{t}, t_L^m)$ ensures the existence of an intersection between f and g at some point $(\tilde{t}_H^p, \tilde{t}_L^p)$. $G_H^p(\tilde{t}_H^p, \tilde{t}_L^p) = G_L^p(\tilde{t}_H^p, \tilde{t}_L^p) = 0$ by definitions of f and g, therefore we have a cut-off equilibrium below the 45 degree line ($\tilde{t}_H^p < \tilde{t}_L^p$).

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