All about priorities: no school choice under the presence of bad schools

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Abstract

When school choice is implemented it is often implied that parents' preferences will affect the school their children attend. The key aspects that school choice procedures need to address are overdemands for schools and the options families have if they are rejected from their first choice. Overdemands are usually resolved by priority rules given for residence in the catchment area of the school and other socioeconomic circumstances. We show that if all individuals agree on what the worse schools are, the two most debated mechanisms, the Boston mechanism and the Gale Shapley (DA), will provide an allocation that fully corresponds to those priorities independently of families' listed preferences. Top Trading Cycles, a third proposal presented in the literature but not implemented yet, improves upon the allocation determined by priorities and therefore is the only responding to parents' preferences. Another interpretation of the results is that if the authorities have some preferences over where families should go to school they can implement them fully through setting priorities accordingly and choosing the Boston or DA mechanisms, which are the two most commonly used mechanisms.

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1 Introduction

The importance of allowing for school choice has been greatly emphasized in the literature and in the policy debate. In the past, children were assigned to their neighborhood school automatically. School choice was then restricted to residential choice or Tiebout choice–see Hoxby(2003), Black (1999), Cullen, Jacob and Levitt (2006). A large fraction of OECD countries has expanded choice in various ways in the last two decades. But if we change from a school zone system, where children are systematically allocated to the school in their zone, to a system where there is school choice and residence only gives you priority in case of overdemand for the school in your zone, how will the final allocation change? The presumption is that the final allocation will change and will respond to families' preferences. In this paper we show that if school choice is implemented using the most common mechanisms, Gale Shapley Deferred Acceptance (DA) or the Boston mechanism, the allocation resulting from it may be very close to the one without school choice and where school zones determine placement. In particular, in the case that school choice is implemented through the Gale Shapley Deferred Acceptance mechanism, the presence of a school that all families agree is slightly worse than the rest of the schools, will force the final allocation resulting from the dominant strategy equilibrium to collapse to all children being allocated to the school in their zone, as if there was no school choice. In the case of the Boston mechanism, we show that for any preference profile there exists a quality level for the worse school for each family for which the unique Nash Equilibrium leads to all children applying and being allocated to the school in their zone. In other words, we show that the implementation of school choice does not guarantee that the final allocation is shaped by families' preferences. Instead, priorities, which in principle should only help breaking ties, almost fully determine the allocation. Authorities then can determine the allocation of children to school through picking priorities and implementing BOS or DA.

The mechanism design literature on school choice has studied the problem of allocating students to a set of schools. This constitutes what the literature refers to as a two sided matching problem, but with the special feature that schools, in this case one of the two sides of the market, often cannot express their preferences over students. School preferences are substituted by priority orders determined by the central administration according to mainly the distance from the families' residence to the school and the existence of siblings in the school. These processes usually work as follows: families submit a list with a raking for the different schools available. Then a set of rules determine how these preferences determine the final allocation together with the priority rules. These priorities are taken as given by the school choice literature and they are viewed as a constraint that the mechanism should respect. The literature has focused on how the set of norms can better allocate children to schools given the submitted preferences, taking priorities as given.

The literature has emphasized different properties of the rules characterizing the mechanism. A first property is for the mechanism to provide incentives to reveal true preferences, what is referred to as the mechanism being strategy proof. The Boston mechanism, one of the most widely used but also criticized mechanisms, lacks this property. This implies that families can get a better allocation by stating a different ranking than their true preferences. Alternative mechanisms, such as the Gale Shapley (DA) and the Top Trading Cycles (TTC), which will be described later in the text, do have this property and therefore elicit true preferences. This greatly simplifies matters for the families. Efficiency is another important property, in this case defined as Pareto efficiency of the final allocation (there does not exist another allocation that makes a student better off without harming another student). DA is not efficient but TTC is. On the other hand DA is valued because it never violates the ranking established by priorities, a property called stability, because it helps the mechanism to survive judicial processes. No mechanism satisfies the three properties simultaneously and so the choice is between truth telling and stability, offered by DA, or truth telling and efficiency, offered by TTC. DA has actually been adopted in cities like New York and Boston, substituting the formerly imposed mechanism that now receives the name of the Boston Mechanism. So both DA and Boston, or a combination of the two (see Chen and Kesten (2011)) are the most debated alternatives (Abdulkadiroğlu and Sönmez, 2003; Abdulkadiroğlu, Pathak, Roth and Sönmez, 2006; Ergin and Sönmez, 2006; Miralles, 2008; Pathak and Sönmez, 2008 and forthcoming; Abdulkadiroğlu, Che and Yasuda, 2011).

This paper suggests that the choice between these two mechanisms may be less important given that in both cases the main determinants of the final allocation of students are the priority rules. Similarly to Miralles (2008), Abdulkadiroglu, Che and Yasuda (2011), we follow Auman (1964) and assume that there is a continuum of individuals to be allocated to a finite number of goods, in this case seats in schools. We also simplify the model by assuming that families have priority for one school and the number of seats the school has coincides with the number of children that have highest priority. We then show that under the presence of worse schools the final allocation in both the Boston and the DA mechanism is determined by the priority order, that is, children are assigned to the schools for which they have highest priority. The reasons are different, but both very plausible. In the case of BOS, if each family considers one of the schools in the system sufficiently bad, they will not risk applying for any other school than the school for which it has highest priority. Calsamiglia and Güell (2012) exploit a change in neighborhood design in Barcelona to show that a substantial fraction of families do apply for their neighborhood school because of its safer properties under the Boston Mechanism. Our results rationalize such empirical observation. Under DA we show that the presence of a school or of a set of schools that all families agree are slightly worse than the remaining schools, that is, all families have them in the bottom of their preference rank, will lead to children being allocated to the school for which they have highest priority. This happens despite of the fact that families submit their true list of preferences. If the we did not assume a continuum of families but discrete, the result would say that the set of children that are not assigned to their neighborhood school goes to zero as the number of children gets large. The intuition for this results is that for two children to swap seats it has to be the case that both win the lottery when considered for a school for which they do not have priority for. But if there are schools that all families want to move away from, that guarantees that all student living in the neighborhood of those schools will apply to any other school, reducing chances for all families of entering any of the schools for which they do not have priority for. Chances would go to zero in the discrete case as the number of children gets large, and is zero in the continuous case.[1]

This paper suggests that the debate should not be on whether Boston or DA, since both lead to allocations that differ little from the case where there is no choice because of the existence of priorities. Importantly it states that with these two mechanisms priorities limit the extent to which families preferences determine the final allocation. But these priorities are determined by the central authorities, just in the same way that the rules respecting those priorities and assigning students to schools are.

This paper emphasizes that the two most debated and used mechanism in the literature and in the policy debate may both be very limited in their capacity of allocating children according to their preferences whenever there exists priorities for children to certain schools to break ties. TTC, mainly ignored in actual applications, responds less to priorities and more to parents' preferences.

The next section presents the baseline model and its results. Section 3 discusses and extends the model in several frontlines: finite economies, strict residential priorities, relaxations in the priority structure and outside options, and the political economy of the priority structure. The last section concludes, with and appendix being used for long proofs.

2 The Model

A school choice problem (Abdulkadiroğlu and Sönmez, 2003) is defined by a set of schools and a set of students, each of which has to be assigned a seat at not more than one of the schools. Each student is assumed to have strict preferences over the schools and the option of remaining unassigned. Each school is endowed with a strict priority ordering over the students and a fixed capacity of seats.

An outcome of a school choice problem is a *matching*, that is, an assignment of school seats to students such that each student is assigned at most one seat and each school receives no more students than its capacity.

We have a set of schools, divided into two subsets: the good (G) and the bad schools. There are $J \geq 2$ good schools indexed by $j \in G$. All the bad schools are identical to all agents so WLOG we assume there is a unique bad school called the worst school and indexed by w.[2] The capacity of each school $i \in G \cup \{w\}$ is denoted by $\eta_i > 0$ and the sum of all capacities is 1. Let $\eta \equiv (\eta_1, ..., \eta_J, \eta_w)$ be the vector of school capacities.

There is a continuum of students $x \in X = [0, 1]^J$ with a total mass of 1, endowed with the Lebesgue measure λ. Each of them is to be assigned to exactly one school.

[1]This result is true for any stable mechanism, a property that has been greatly emphasized in the literature– see Roth (2008).

[2]We could allow for more than one bad school, making the proportion of "bad" slots plausible. What is important is that there is "low conflict" with regard to the choice among bad schools, in the sense that each agent can at least ensure being assigned at her best among those schools. The valuation of that school is then normalized to v_w, going back to our model.

Students' preferences are defined by a measurable valuation function $v : X \to \mathbb{R}_+^J \times \mathbb{R}_-$, where the last component is the valuation for the worst school. We assume that essentially no student is indifferent among any two schools: $\lambda(\{x \in X : v_i(x) = v_j(x)$, some $i, j \in G \cup \{w\}\}) = 0$. Let V be the set of all such v.

An assignment is a measurable function $\mu : X \to G \cup \{w\}$. $\mu(x)$ denotes the school to which student x is assigned. An assignment μ is feasible if $\lambda(\{x \in X : \mu(x) = j\}) = \eta_j$, $\forall j \in G$

A random assignment is a function $q : X \to \Delta^J$ whose j-th element $q_j(x)$ is the probability that student x is assigned to school j. The random assignment q is feasible if capacities are attained in expected terms, $\int_{x \in X} q_j(x) d\lambda = \eta_j$, $\forall j \in G$.[3] A student x evaluates her random assignment according to her expected utility $q(x) \cdot v(x)$. Let $Q = Q(\eta)$ denote the set of all feasible random assignments given the capacities η.

A *binary priority structure* is a measurable function $\pi : X \to G \cup \{w\}$ such that $\lambda(\{x \in X : \pi(x) = j\}) = \eta_j$, $\forall j \in G$. $\pi(x)$ denotes the school at which student x has priority.

Definition 1 *We say that a random assignment q is* driven by priorities π *if it collapses to an assignment $\mu = \pi$.*

An *economy* is a tuple of preferences, capacities and priorities (v, η, π). Let $\sigma : X \to \Pi$ denote a *pure strategy profile*, where Π is the set containing all $((J + 1)!)$ permutations (orders) over the vector $(1, ..., J, w)$ and $\sigma_n(x)$ is the school that student x puts in n-th position. Let Σ denote the set of all measurable σ . A game is a functional $\Gamma : \Sigma \times \pi \to Q$ that determines a random matching for every strategy profile σ, where $\Gamma(\sigma, \pi)(x) = q(x)$. Abusing notation, we use (s, σ) to indicate that some agent (say x) uses strategy s and the rest of agents choose their strategies according to the strategy profile σ. We focus attention on equilibria in pure strategies.

Definition 2 *A Nash Equilibrium of the game generated by a given mechanism is a strategy profile $\sigma^* = \sigma^*(v, \eta, \pi)$ such that*
$\Gamma((\sigma^*(x), \sigma^*), \pi) \cdot v(x) \geq \Gamma((s, \sigma^*), \pi) \cdot v(x)$ *for all $s \in \Pi$ and for all x a.s.*

Definition 3 *A Dominant Strategy Equilibrium (DSE) is a strategy profile $\sigma^* = \sigma^*(v, \eta, \pi)$ such that*
$\Gamma((\sigma^*(x), \sigma), \pi) \cdot v(x) \geq \Gamma((s, \sigma), \pi) \cdot v(x)$ *for all $s \in \Pi$ and all $\sigma \in \Sigma$, and for all x a.s.*

The *outcome* of a NE (or DSE) σ^* is the random assignment $q^* = \Gamma(\sigma^*, \pi)$.

[3] There is no Birkhoff-von Neumann Theorem that applies to all continuum economies. However, for the purposes of the present paper, that is not a major concern. Since both the strategy space and the range of π are finite and given that the strategy profile is measurable, one can partition X into a finite number of subsets according to their played strategies s and to their priority status π. Each subset could be treated as an individual with mass $\lambda_{s\pi}$, and the Birkhoff-von Neumann Theorem would apply to this "finite" economy.

2.1 Deferred Acceptance (DA)

The algorithm that characterizes the DA can be summarized as follows:

- In every round, each applicant applies to the highest school in its submitted list that has not rejected it yet.
- For every round k, $k \geq 1$: Each school tentatively assigns seats to the students that apply to it or that were preaccepted in the previous round following its priority order (breaking ties through a fair lottery). When the school capacity is attained the school rejects any remaining students that apply to it in that round.
- The DA mechanism terminates when no student is rejected. The tentative matching becomes final.

Proposition 1 *Let the DA_π be the random assignment generated by the DA algorithm with priorities π and a fair tie-breaking rule. The DSE of DA_π provides a degenerate random assignment driven by priorities.*

Proof. Consider any fair lottery outcome $m : X \to [0, 1]$ that is used to break ties in the increasing order. Under the DSE, DA provides an assignment that is stable according to the students' preferences and the strict priorities implied by π and m. Let $S_j = \{x \in X : \pi(x) = j\}$ denote the set of students with priority at school j, and let $S_G = \bigcup_{j \in G} S_j$ denote the set of students with priority at some good school. Given that all students in S_G have priority at some school $j \in S_G$ and that all of them put school w in last position as part of the DSE, none of them are finally assigned to w. Hence, all students in $S_w = X \backslash S_G$ are assigned to w. Suppose that a positive-measured set of students in S_G are assigned to schools other than the ones at which they have priority. Let $\bar{m}$ be the highest lottery number in that set. Since the lottery is fair, there must be an $\bar{m}\eta_w$ measure of students in S_w with a lottery number below that threshold. This would lead to a violation of stability (some of the latter students would have strict priority over some of the former), a contradiction. ■

This result is very strong and robust. A common worst school constitutes a sufficient condition for an assignment driven by priorities as a unique prediction, regardless of how preferences are distributed among the good schools. From the proof one could infer that stability is the key element. Indeed, for each fair lottery outcome, the unique ex-post stable assignment is the one driven by priorities.

Such a strong result depends on the continuum assumption, and it is clear that the last step of the proof fails to be true in finite economies. Instead, there is a positive probability that some of the students with highest priority for w obtain lower lottery numbers than some student with priority in a good school who is assigned to a school other than the one where she has priority. This would allow students with good schools to indirectly swap schools. However, it is also apparent that, as the total number of slots grow large at each school, those chances converge to zero. In finitely large economies, our result constitutes a rather precise approximation to what one could actually expect.[4]

We see next that the same (limit) outcome obtains with a mechanism that does not provide ex-post stable assignment.

2.2 Boston Mechanism (BOS)

The algorithm that characterizes the BOS mechanism can be summarized as follows:

- In every round, each applicant applies to the highest school in its submitted list that has not rejected it yet.
- For every round k, $k \geq 1$: Each school assigns seats to the students that apply to it following its priority order, and breaking ties through a random lottery. If the school capacity is or was attained, the school rejects any remaining students that point to it.
- The Boston mechanism terminates when all students have been assigned a school.

Considering BOS, there is a Nash equilibrium that obtains the assignment that is completely driven by priorities. That this outcome is unique depends on the parameters, primarily on $v_w(\cdot)$. In words, the proposition states that for any given preferences if the valuation of the worse school is low enough then the set of individuals applying for a school different then the high priority school can be made arbitrarily small. The higher the capacity of the bad school, and the lower its valuation, the lower the maximum proportion of students with priority at a good school that put another school in first position in a NE. In particular, if preferences are bounded we can find a valuation for the worse school such that the set of families applying for a different school than the highest priority school is zero.

Proposition 2 *For all $v \in V$ and for each $\varepsilon \in (0, \eta_w/2)$ there is $v_w^\varepsilon \leq 0$ such that if $v_w(x) < v_w^\varepsilon$ for all $x \in X$, then any NE $\sigma^* = \sigma^*(v, \eta, \pi)$ in BOS is such that $\lambda(\{x \in S_G : \sigma_1^*(x) \neq \pi(x)\}) < \varepsilon$.*

In particular, if V is upper-bounded, then for all $v \in V$ there is $v_w^0 \leq 0$ such that if $v_w(x) < v_w^0$ for all $x \in X$, then the (essentially) unique NE $\sigma^* = \sigma^*(v, \eta, \pi)$ in BOS satisfies $\lambda(\{x \in S_G : \sigma_1^*(x) \neq \pi(x)\}) = 0$.

Proof. Let q^v denote the outcome of a BOS-NE $\sigma_v^* = \sigma^*(v, \eta, \pi)$. As in the previous proof, let $S_j = \{x \in X : \pi(x) = j\}$ denote the set of students with priority at school j, and let $S_G = \bigcup_{j \in G} S_j$ denote the set of students with priority at some good school. Consider a student x who is best-responding to σ^*. Consider any two strategies $\tilde{s}$ and s' such

[4]See Azevedo and Leshno (2011) for a result on the convergence of the allocation in the discrete to allocation in the continuum.

that the probabilities to end up in the worst school are $\tilde{q}_w(x) \equiv BOS_w((\tilde{s}, \sigma_v^*), \pi)(x) < BOS_w((s', \sigma_v^*), \pi)(x) \equiv q'_w(x)$. Since the student is maximizing the expected value, for a low enough valuation of the worse school $v_w(x)$ student x chooses strategy $\tilde{s}$. Generalizing, any student x chooses among the strategies yielding minimal probability to be assigned to the worst school if $v_w(x)$ is low enough. Since all students in S_G have a strategy that reduces the probability of being assigned to the worst school to zero, we have that there exists an v_w^G such that if $v_w(x) < v_w^G$ for all $x \in S_G$ in equilibrium $\lambda(\{x \in S_G : q_w^v(x) > 0\}) < \varepsilon$. Let $\bar{q}_w$ be the minimum probability (depending on the played strategy) of being assigned to the worst school for any student $x \in S_w$. By the previous argument, $\eta_w - \lambda(\{x \in S_w : q_w^v(x) = \bar{q}_w\}) < \varepsilon$ for $v_w(x)$ low enough ($v_w(x) < v_w^{G'}$) for all $x \in S_w$. Select $v_w^\varepsilon = \min\{v_w^G, v_w^{G'}\}$, and consider $v_w(x) < v_w^\varepsilon$ for all $x \in X$. By feasibility, we have $1 - \bar{q}_w \leq \frac{\lambda(\{x \in S_G : q_w^v(x) > 0\})}{\lambda(\{x \in S_w : q_w^v(x) = \bar{q}_w\})} < \frac{\varepsilon}{\eta_w - \varepsilon}$, thus $\bar{q}_w > 0$. That is, $\lambda(\{x \in X : \sigma_1^*(x) = j\}) \geq \eta_j$ for all $j \in G$ (otherwise $\bar{q}_w$ would be zero by putting an underdemanded school in first position). But then, for any student $x \in S_G$, the only strategies producing $q_w^v(x) = 0$ are those such that $\sigma_1(x) = \pi(x)$. Hence $\lambda(\{x \in S_G : \sigma_1^*(x) \neq \pi(x)\}) = \lambda(\{x \in S_G : q_w^v(x) > 0\}) < \varepsilon$.

The proof for the second part follows a similar argument. Given that valuations are bounded, we have both $\lambda(\{x \in S_G : q_w^v(x) > 0\}) = 0$ and $\eta_w - \lambda(\{x \in S_w : q_w^v(x) = \bar{q}_w\}) = 0$ for $v_w(x)$ low enough for all $x \in X$ ($v_w(x) < v_w^0$). It follows by feasibility that $\bar{q}_w = 1$, and therefore $\lambda(\{x \in S_G : \sigma_1^*(x) \neq \pi(x)\}) = \lambda(\{x \in S_G : q_w^v(x) > 0\})$, which is equal to zero. ■

Example 1 ***I.I.D. bounded valuations.*** *Consider a valuation function such that for any $x \in X$ and any $j \in G$, $v_j(x)$ is independently drawn from an atomless distribution F with support $I \subset [0, 1]$. School capacities are η_w for the bad school and $\frac{1-\eta_w}{J}$ for each of the J good schools. Let the priority structure π be the result of a fair lottery ($v(x)$ and $\pi(x)$ are uncorrelated). If $v_w(x) < -\frac{1-\eta_w}{\eta_w}$ for all $x \in X$ there is a unique NE σ^* in BOS such that $\lambda(\{x \in S_G : \sigma_1^*(x) \neq \pi(x)\}) = 0$.*

The valuation of the bad school controls the relative valuations between schools. That is, it controls how important being assigned to any school other than the worst school is. Indeed, a student x chooses between two schools i and j by comparing the quotient of valuations $\frac{v_i(x) - v_w}{v_j(x) - v_w}$ to the quotient of assignment probabilities. This quotient converges to 1 as $-v_w$ grows big enough. From this example we also conjecture that the critical value v_w^0 depends negatively on the capacity of the bad school. The capacity of the bad school approximates the risk of attending the worst school for a student that departs from applying first to the school that gives her priority. The argument in BOS is not about stability but about risk avoidance.

Our result regarding the Boston Mechanism also holds in finite economies. Moreover, it not only a limit result. When the valuation of the bad school is low enough, every student with priority in a good school optimally chooses to ensure not being assigned to w. By applying to a school other than $\pi(x)$ as a first choice, student x runs into a risk of being rejected whereas a student with priority in the worst school takes a slot in a good school, meaning that a student rejected from the first option ends up being assigned to w. Thus every agent with priority in a good school will optimally put $\pi(x)$ as her first choice.

2.3 Top Trading Cycles

The TTC algorithm is summarized as follows. A fair lottery $m : X \to [0, 1]$ is run to break ties in the increasing order, and a lexicographic strict priority ordering is constructed for each school. In this lexicographic priority, the weak priority structure π is used as the primary criterion and the tie-breaking lottery as secondary criterion.

- Each school points at its preferred student among the ones not yet assigned, using the lexicographic strict priority as a preference criterion.
- Each student points at her most-preferred school among the ones with available slots.
- A cycle is found in which a school points at a student who points at a school which points at a student... who finally points at the first school. This is always possible since we have a finite number of schools.
- In the cycle we find, each student is assigned to the school she pointed at. These students and their assigned slots are removed from the algorithm.
- We repeat all the steps until no students keep unassigned.

We illustrate that a student-Pareto improving assignment is obtained through the DSE in TTC. It is true that there is another NE (each student in S_G putting her priority-giving school in first position) that provides the assignment that is driven by priorities. However, this is a NE in dominated strategies, in which case we select the DSE (truth-telling) as a natural prediction.

Proposition 3 *The DSE in TTC obtains as an outcome a lottery over ex-post student-efficient assignments in which the students with priority in the worst school attend the worst school.*

We do not provide a proof since it follows a standard argument. The conclusion is that TTC is good for ex ante student-efficiency purposes. It Pareto-dominates the unique DSE assignment in DA and the limit NE assignment in BOS.

Moreover, under a binary priority structure, any final assignment under the DSE in TTC is *stable* according to the (weak) binary priority structure π. On the basis of this priority structure, no student with priority in the worst school has a valid claim against the assignment. And every student with priority at a good school j is assigned either to j or to a preferred school.

The relevant difference between BOS and DA on the one side, and TTC on the other, is that TTC manages to block the interferences that students with priority in the worst school could exert over the trading of property (priority) rights among the students with priority in the good school.[5] Although that feature discriminates the unfortunate students who do not have priority at any good school, we have seen that the absence of such discrimination has the only effect of blocking the aforementioned trading.

Given the priority structure π, TTC coincides with the solution suggested by Ergin and Erdil (2008) to the problem of DA under weak priorities. In the general case with any given structure of weak priorities, the trading cycles preserving stability that these authors suggest are supported by a Nash Equilibrium, yet strategy-proofness cannot be guaranteed. The priority structure we analyze in this paper is a fortunate case in which the solution they suggest is supported by dominant strategies.

3 A mechanism design approach to priorities

In the theoretical literature on school choice, it is commonly assumed that the central authority is "benevolent" in the sense that it chooses the assignment mechanism in order to meet some efficiency goals ("to maximize students' choice") while respecting some priority rules that are exogenously given. What if that were not the case? It is rather plausible that the authority has its own goals concerning the assignment while maintaining an "illusion of choice". For instance, to minimize the school busing service costs, or to satisfy the requests of an important group of voters. Let us consider an authority's *bliss assignment* $\beta : X \to G \cup \{w\}$. The authority's alternative problem would be to set priorities so as to minimize some distance measure between its outcome and the bliss assignment, taking the assignment mechanism (either DA or BOS) as a constraint. If that were the problem to solve, a solution (or limit solution in BOS for $v_w(\cdot)$ low enough) would be to set the binary priority structure $\pi = \beta$. This solution is indeed *simple* in the sense that it does not depend on any information on students' preferences, apart from the fact that there is a bad school that everyone dislikes.

We are not stating, however, that this is precisely what the school authorities do. The authority might have an objective function that averages a bliss assignment (induced by groups of pressure, busing cost minimization etc.) with a concern about satisfying parents' preferences. This could explain a case that is largely ignored in the theoretical literature: in Boston schools, only half (if any) of available seats are assigned using walking-zone priority as a priority criterion.[6] This is interesting because the Boston case has been one of the most discussed cases in the school choice literature.

Yet this observation supports our main ideas in our paper: the authority may be aware that residence priorities could drive parents' preferences off the assignment. Other authorities such as the San Francisco Unified School District are aware of the effects of residence priority on the actual choice. For that reason, living in an area with "bad

[5]There is an ethical concern about trading with priorities. However, according to Abdulkadiroğlu and Che (2010), TTC can be characterized as a series of properties that do not involve this trading. Conceiving TTC as a "trading process" is just a simplification that helps us understand the final assignment.

[6]http://www.bostonpublicschools.org/assignment

schools" (the lowest 20% percentile of average test scores) counts as a lexicographically higher priority criterion than residence priority.[7] More recently, the Madrid Regional School Authority has ruled out all residential priorities. The Barcelona School Authority has also recently widened the "residential priority zone" of all applicants after a 2008 - 2011 period in which each student had a rather limited number of schools giving her residence priority.

4 Extensions

We now show that the main results presented in section 2 are robust to relaxing or extending the assumptions in the model.

4.1 Finite economies

A fair question involves the robustness of our results if the number of agents is finite. If we consider the Boston mechanism, we can readily state that proposition 2 holds. The finiteness assumption makes the result actually easier. Since we are analyzing Nash equilibria in which all students know all students' preferences, the valuation space could be considered as bounded. Then there must exist a value v_w^0 such that there is only one NE giving an assignment that is completely driven by priorities, if all students' valuations for the worst school lie below v_w^0.

Concerning DA, the result in proposition 1 might not hold. Chances are that *all* the students in S_G aiming to a school other than the priority-giving school obtain better lottery numbers than *all* the students in S_w. Such an event, however, happens with very low probability in big economies, indeed with a high speed of convergence. Consider two numbers $\gamma, \eta_w \in \mathbb{N}$. Let us construct a sequence with $n = 1, 2...$ in which a set X'_n contains γn i.i.d. draws from the uniform distribution while the set S_w^n contains $\eta_w n$ i.i.d. draws. It represents an economy that keeps the ratio of draws in X'_n (i.e. students assigned to a school other than the priority-giving school) per draws in S_w^n (i.e. students with no priority at any good school) constant in γ/η_w while it grows. The probability that the maximum number in X'_n is lower than the minimum number in S_w^n is $(\gamma n)!(\eta_w n)!/[(\gamma + \eta_w)n]! = \begin{pmatrix} (\gamma + \eta_w)n \\ \gamma n \end{pmatrix}^{-1}$, which goes to zero factorially fast as n grows.

4.2 Relaxing assumptions on preferences

It is necessary to assume that everyone dislikes the worst school? Particularly, one could think that frequently a proportion of students (parents) in S_w actually likes the worst school. Motives are apparent: distance to the household is an important factor in deciding which school is good for the child, even in unpopular neighborhoods of for unpopular schools. Fortunately, the model could accommodate these considerations while keeping the

[7] Source: http://www.sfusd.edu.

main results unchanged. In fact, the only requirement is that a positive mass of students in S_w has the worst school as the school they prefer the less. With this, propositions 1 and 2 hold, since for the proofs to hold we just need a positive mass of students in S_w to be interested in attending any school other than the worst school.

As for the preferences of the students (parents) in S_G, we analyze DA and BOS separately. From the proof of proposition 1, we see that in DA we do not need everyone ranking the worst school in last position. It is enough if the worst school lies below the priority-giving school according to the ordinal preferences of each student with priority at a good school. Under this condition, because any student in S_G will be assigned to a school that is at least as good as the priority-giving school, only students in S_w are assigned to the worst school. The stability argument of the proof can again be used to preclude students from attending a school different from the priority-giving one. Regarding BOS, we do not need the assumption that the worst school is actually the worst one for everyone. However, we need to identify a common school w that is bad enough for everyone, even if that is not the worst school for everyone. That is, we can allow other schools to be worse than w, but w must be bad enough for everyone, in order to hold the results of proposition 2.

4.3 A strict residential priority structure

One could have considered a model in which the residential priority ordering gives an essentially strict priority structure by means of measuring distance to the school, as done in Belgium. Consider the following *linear city* model in which agents are located uniformly along the [0,1] interval. Let $L(x)$ indicate the location of agent x. Schools are located in the following way: school 1 is located at $L_1 \equiv \eta_1/2$, each school $j \in \{2, ..., J\}$ is located at $L_j \equiv \sum_{i<j} \eta_i + \eta_j/2$, and finally school w is located at $L_w \equiv \sum_{i \in G} \eta_i + \eta_w/2$. Other setups including a circular city or a different location for the worst school could be used and the results would remain basically unchanged. The distance of agent x to school $i \in G \cup \{w\}$ is simply $d_i(x) \equiv |L(x) - L_i|$, and an agent x has priority over agent y at school i if $d_i(x) < d_i(y)$ (see that there could be ties yet on a zero-measured set of cases). We would like to characterize the stable assignment given this priority structure. Accordingly, each school j has a "zone of influence" $(\sum_{i<j} \eta_i, \sum_{i \leq j} \eta_i)$ such that any student located there has sure assignment in that school if wished, in all three mechanisms here studied. In that sense, our previous weak priority structure π could be derived as follows: $\pi(x) = j \iff L(x) \in (\sum_{i<j} \eta_i, \sum_{i \leq j} \eta_i)$. The partition $\{S_j\}_{j \in G \cup \{w\}}$ is again constructed from the weak priority structure $\pi : S_j = \{x \in X : L(x) \in (\sum_{i<j} \eta_i, \sum_{i \leq j} \eta_i)\}$.

We impose the following assumption. Consider the following notation: for $k \neq j$ and $\varepsilon > 0$, $\varphi^{\varepsilon}_{kj} = \lambda(\{x \in X : L(x) \in (\sum_{i<j} \eta_i, \sum_{i<j} \eta_i + \varepsilon), v_k(x) > v_j(x)\})$. $\varphi^{\varepsilon}_{kj}$ is the measure of students located at least an $\varepsilon-$close to the left-hand side frontier of school j's zone of influence who prefer school k to j. We assume that $\forall \varepsilon > 0$ and for any $j \in \{2, ..., J\}$, $\varphi^{\varepsilon}_{(j-1)j} > 0$. We refer it as a *next-to-border preference variety assumption.* A positive measure of students in the zone of influence of a good school j and close to the zone of influence of $j-1$ prefers $j-1$ to j. It is simply an assumption of imperfect correlation between proximity to each school and ordinal preferences across them.

Proposition 4 *In the linear city with next-to-border preference variety, the essentially unique stable assignment is $\mu = \pi$.*

Proof. Let μ denote a stable assignment in this economy. A first observation is $\lambda(\{x \in S_w : \mu(x) \neq w\}) = 0$, or else by feasibility a positive-measured set of students in S_G is assigned to w, violating stability since w is the least-preferred school. A second observation is $\lambda(\{x \in S_J : \mu(x) \neq J\}) = 0$. Otherwise, $\sup\{d_J(x) : x \notin S_J, \mu(x) = J\} > \inf\{d_J(x) : x \in S_w\}$, again violating stability (since all agents in S_w prefer J to w). Then the next induction argument completes the proof: if for some $j \in \{2, ..., J\}$ we have $\lambda(\{x \in S_j : \mu(x) \neq j\}) = 0$, we also have $\lambda(\{x \in S_{j-1} : \mu(x) \neq j-1\}) = 0$. Otherwise, $\sup\{d_{j-1}(x) : x \notin S_{j-1}, \mu(x) = j-1\} > \inf\{d_{j-1}(x) : x \in S_j, v_{j-1}(x) > v_j(x)\}$ (here we make use of the next-to-border preference variety assumption), again violating stability. ■

Corollary 1 *In the linear city with next-to-border preference variety, both the DSE in DA and the (essentially) unique NE in BOS lead to an outcome that is completely driven by priorities.*

The corollary stems from the fact that, under strict priorities, both DA (in DSE) and BOS (in NE) generate only stable assignments (Ergin and Sönmez, 2006). The analysis here undertaken constitutes a particularly relevant result since some school authorities have indeed decided to switch their residence priority criteria from a discrete to a continuum one. As we have seen, the problem of ignoring students' preferences remains the same or even worse. For instance, the bad result in BOS is not only a limit result anymore.

4.4 Other priority structures

One of the concerns about the model is the way the priority structure is conceived. The binary priority structure could be justified insofar as it constitutes an approximation to a common priority criterion in school choice, namely district priority or walking zone priority. However, we acknowledge that the priority structure in school choice may contain more ladders. For instance, the presence of a siblings at the school, or having a low family income, constitute facts that give the students a higher priority level. The set of possibilities is much richer than what our theoretical model involve. Some discussion is in order to consider other cases.

For instance, we could consider a discriminatory trinomial priority structure $\pi^3 : X \to \{0, 1, 2\}^{J+1}$ constructed from the binary structure π in the following way for each school j: $\pi^3_j(x) = 2$ if $\pi(x) = j$, $\pi^3_j(x) = 1$ if $\pi(x) \in G\backslash\{j\}$, $\pi^3_j(x) = 0$ otherwise. Under π^3, it can be seen that DA's DSE outcome Pareto-improves the outcome implied by π (obtaining a lottery over ex-post student-efficient assignments in which the students with higher priority in the worst school attend the worst school). The reason is that the students with priority in the worst school cannot block the trading over property (priority) rights among the students with priority in a good school. Implicit swaps between individuals from good schools can only happen if individuals from good schools have higher priority for other good schools than students with highest priority in the worst school do. For example, if points for socioeconomic circumstances are given, those will most likely be given to students with higher priority for the worst school. If that is the case, the original results would still hold true, since students with priority in the worst school would be ahead of students with priority in a good school who are applying for another good school. So the violation of the original results would require that absolutely lowest priority is given to individuals with priority at the worst school and that individuals with priority at the good schools are given some intermediary priority for other good schools, explicitly isolating those individuals with priority at the worst school. This emphasizes further that the problem with DA is that allowing for swaps involves competing for the seats with students that will systematically be interested in those schools and may have the same chances of entering. The fact that no student wants to end up in a bad school insures no possible exchanges from the allocation determined by highest priorities.

As for BOS, Proposition 2 does not longer hold in all cases. As a counterexample, consider the preferences of example 1 (i.i.d. valuations for the good schools) and the priority structure π^3 as described above. There is a Nash equilibrium where each student with high priority in a good school puts her most preferred school in first position. This is possible since the students with priority in the worst school cannot compete against the rest of students in the first assignment round. The latter examples illustrate that an enrichment of the priority structure can improve the efficiency of the assignment. The price to pay is, as seen, a higher discrimination of students that were already discriminated.

Instead of adding more priority levels to the priority structure, one could wonder about the robustness of our results to a binary priority structure that gives agents priority at possibly more than one good school. Then π could be understood as set-valued, $\pi : X \to 2^{G\cup\{w\}}$. Let us assume that the students in S_w still cannot have priority at any good school. In what sense would our results be modified? As for DA, this is what would happen: In any final assignment μ arising from the DSE in DA, for any $x \in X$ we have $\mu(x) \in \pi(x)$. That is, students may have some "choice", but only among the schools giving them priority. Otherwise stability would be violated following the lines of the proof of proposition 1. Regarding BOS, basically there would be minor changes in proposition 2. Students would have avoiding the worst school as the main goal, and any NE would have $\sigma_1(x) \in \pi(x)$ for any $x \in S_G$ in a way that $\lambda(\{x \in X : \sigma_1(x) = j\}) = \eta_j$ for any good school j.

4.5 Outside Options

A natural question regarding this model is to what extent the results may change if we allow for outside options (home schooling and private schools). As a matter of notation we use the function $o : x \to \mathbb{R}$ for the valuation profile of the outside option. We comment on DA and BOS separately.

In DA, let us assume that a measure γ of students in S_G have an outside option that is actually preferred to the school that gives them priority. We find an upper bound on the mass of students that could be assigned to a school other than the priority-giving school. First, notice that at most a measure γ of students in S_w would finally occupy slots in good schools.[8] Then the lowest lottery number among the students in S_w who end up assigned to the worst school cannot be higher than γ/η_w. Then only students in S_G with a lottery number lower than γ/η_w could be assigned to a school different than the one giving them priority. The following remark follows.

Remark: Let $\gamma \equiv \lambda(\{x \in S_G : o(x) > v_{\pi(x)}(x)\}) < \eta_w$ and let X' denote the set of students in S_G that are assigned to a school different from the one giving them priority under DA. Then $\lambda(X') \leq \gamma(1-\eta_w)/\eta_w$.

As for BOS, the presence of outside options do not modify our results as long as these options are not valuable enough. Indeed, proposition 2 would hold as long as $\max\{o(x), v_w(x)\}$ (instead of only $v_w(x)$) is low enough for every $x \in X$ as defined in the proposition.[9]

5 Conclusions

This paper shows how the existence of bad schools together with the existence of priorities can completely determine the allocation of children to schools when either the Boston or Gale-Shapley's Deferred Acceptance mechanisms are used. In some sense, then, given that the authorities have only implemented combinations of one or the other as long as there is a set of schools that are considered worse than the rest of the schools, families have basically little influence on the allocation. Students generally are placed where the authorities give them highest priority. Alternative mechanisms that do not put a high risk on the first choice or that are not stable would improve the capacity of the final allocation to be shaped by parents' preferences. TTC would seem like the natural candidate.

6 Appendix

Proof of Example 1

Proof. That the aforementioned equilibrium exists can be easily proven and we omit it here (see the argument preceding proposition 1). We show uniqueness of σ^*.

First, there is no strategy $s_j \in \Sigma$ guaranteeing sure assignment to a school $j \in G$ for students who do not have priority there. Given that preferences among good schools are uniformly distributed, a mass of at least $1/J$ students would be using such strategy s_j. But then the feasibility constraint for school j would be violated since each good school has a capacity $\frac{1-\eta_w}{J} < \frac{1}{J}$. It follows that all the slots of the good schools are given in the first assignment round.

[8] This is quite a generous upper bound since it is equivalent to assume that there is a mass γ of good-school slots "in excess" (that is, with no assigned priority-holder). The remark that follows can therefore be applied to excess capacity cases as stated before.

[9] Calsamiglia, Miralles and Martinez-Mora (2012) analyze extensively these mechanisms under the presence of private schools.

Let us use $q_j < 1$ for the probability of being assigned to school $j \in G$ for a student $x \in X$ such that $\sigma_1^*(x) = j \neq \pi(x)$. We show that q_j is the same for all $j \in G$. Let us use $\alpha_j = \frac{\lambda(\{x \in S_j : \sigma_1^*(x) \neq j\})}{(1-\eta_w)/J}$ for the proportion of students with priority at school $j \in G \cup \{w\}$ who put another school in first position (notice that $\alpha_w = 1$ unless $q_j > 0$ for any $j \in G$). We also denote $\alpha_{ij} = \frac{\lambda(\{x \in S_i : \sigma_1^*(x) = j \neq i\})}{\alpha_i}$ for the proportion of students with priority at school $i \in G \cup \{w\}$ not putting i in first position, who put another school $j \in G \backslash \{i\}$ in first position. Consider any two good schools i and j such that $q_i \geq q_j$. It follows by the i.i.d. assumption that $\alpha_i \leq \alpha_j$, $\alpha_{ij} \leq \alpha_{ji}$ and $\alpha_{hj} \geq \alpha_{hi}$ for any $h \in G \cup \{w\} \backslash \{i, j\}$. Feasibility implies $q_i \cdot \left(\frac{1-\eta_w}{J} \sum_{h \in G \backslash \{i\}} \alpha_h \alpha_{hi} + \eta_w \alpha_{wi} \right) = \frac{1-\eta_w}{J} \alpha_i$ and $q_j \cdot \left(\frac{1-\eta_w}{J} \sum_{h \in G \backslash \{j\}} \alpha_h \alpha_{hj} + \eta_w \alpha_{wj} \right) = \frac{1-\eta_w}{J} \alpha_j$. This implies $q_i \leq q_j$, thus $q_i = q_j$ as we wanted to show. Having seen that q_j is the same for all $j \in G$, we have $\alpha_{ij} = \frac{1}{J-1}$ for any $i, j \in G$, $\alpha_{wj} = \frac{1}{J}$ for any $j \in G$, and $\alpha_j = \alpha$ for any $j \in G$.

We finally show that $\alpha = 0$ for $v_w(x) < -\frac{1-\eta_w}{\eta_w}$ for all $x \in X$. Assume otherwise that a proportion $\alpha > 0$ of students at each set S_j apply for a school other than j in the first round. Given that preferences among the good schools other than the one giving priority are drawn from the same distribution, this proportion α is evenly split among the remaining good schools $i \neq j$. For the same reason, the students that do not belong to S_G apply in equal proportions to all the good schools. For a good school j, the number of students who apply with priority (and obtain sure placement) is $(1-\alpha)(1-\eta_w)/J$. Thus the remaining slots to allocate among students without priority is $\alpha(1-\eta_w)/J$. The number of applicants without priority is $\frac{\alpha(1-\eta_w)/J}{J-1} \cdot (J-1) + \eta_w/J = [\alpha(1-\eta_w) + \eta_w]/J$, and each one has a chance $q(\alpha, \eta_w) = \frac{\alpha(1-\eta_w)}{\alpha(1-\eta_w)+\eta_w}$ of being accepted at that school. In case of rejection, there are no remaining slots at other good schools and each rejected student ends up in the worst school. In this SNE there is a positive threshold value $\tau(\alpha, F, J)$ verifying

$$q(\alpha, \eta_w) = \tau(\alpha, F, J) \tag{1}$$

A student x with priority at the good school j would apply to the most-preferred good school $i \neq j$ if $\frac{v_j(x)-v_w(x)}{v_i(x)-v_w(x)} \leq \tau(\alpha, F, J)$, and she would apply to school j otherwise. In the SNE α is equal to the probability that $\frac{v_j(x)-v_w(x)}{v_i(x)-v_w(x)} \leq \tau(\alpha, v_w, F, J)$ for some school i different from the one (j) where the student has priority. Denoting the distribution of this variable as $\Psi(F, J)(\cdot)$, we have $\tau(\alpha, F, J) = \Psi(F, J)^{-1}(\alpha)$ (the inverse could be set-valued when $\alpha = 0$, then the condition in equation (1) changes to $q(0, \eta_w) \in \tau(0, F, J)$).

Notice that $q(\alpha, \eta_w) \leq 1 - \eta_w$ for any $\alpha \in (0, 1]$. If for every $x \in X$ we have $v_w(x) < -\frac{1-\eta_w}{\eta_w}$, it follows that $\Psi(F, J)(1-\eta_w) = 0$ (the event $\frac{v_j(x)-v_w(x)}{v_i(x)-v_w(x)} \leq 1 - \eta_w$ has zero probability, since the support of F is the interval $[0, 1]$). Consequently, for any such $v_w(\cdot)$ and any $\alpha \in (0, 1]$ we have $\tau(\alpha, F, J) > 0$ and there is no $\alpha > 0$ meeting equation (1). ■

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