Large vs. Continuum Assignment Economies

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Abstract

Continuum models are often used to study large finite assignment economies. We show that in the canonical assignment problem without transfers, the continuum model can have very different qualitative properties than large finite markets. The problem is driven by the failure of local non-satiation inherent in no-transfer assignment.

Keywords: Random Assignments, Efficiency, Envy-freeness, Convergence Failure, Competitive Equilibrium from Equal Incomes.

1 Introduction

Since Aumann's (1964) pioneering work, it has been widely accepted that continuum economies constitute a valid approximation to big economies where no single agent can have an impact on general market conditions.[1] Following the work of Abdulkadiroglu, Che, and Yasuda (forthcoming), Miralles (2008), Che and Kojima (2010), and Azevedo and Leshno (2013), continuum models became the workhorse models for analyzing large finite matching and assignment economies without transfers. Their usefulness goes beyond theory and extends to recent advances in empirical analysis of these markets (Agarwal and Somaini 2014).

We show that the use of the continuum models as approximations to large finite assignment economies without transfers requires care. We demonstrate it by examining a natural question—what assignments are efficient and fair?—in large finite economies, and by showing that the answer is substantially different

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[1] Continuum approximations offer many methodological advantages. For instance, the standard abuse of the law of large numbers allows us to reduce complex stochastic problems to deterministic ones and the existence of pure-strategy Nash equilibria is assured when the strategy space remains finite (Mas-Colell 1984).

from the answer to the same question in the continuum economy limit. We study economies in which each agents evaluate the utility from a random assignment in line with the expected utility theory and we assume that each agent would like to receive at most one object as in Hylland and Zeckhauser (1979). In the continuum assignment economy limit, Ashlagi and Shi (2014) provided an elegant characterization of efficient and fair assignments: in the full-support economies every such assignment can be implemented in a competitive equilibrium with equal incomes (CEEI); the CEEI mechanism has been first studied in Hylland and Zeckhauser's seminal work.[2]

We show that this elegant characterization fails to be true, even approximately, in large finite economies. To do so we study a sequence of growing finite assignment economies that converge to a full-support continuum economy. In these economies, we construct a sequence of efficient and fair assignments that cannot be supported by competitive equilibria in which agents' budgets (or incomes) are close to equal.

The key to our conterexample is the failure of local non-satiation that is inherent to Hylland and Zeckhauser's assignment economies. In fact, in economies in which agents are locally non-satiated Zhou (1992) shows that all fair and efficient allocations can be implemented via competitive equilibrium from equal incomes not only in the continuum limit but also in large finite economies.[3]

Our note contributes to several strands of the literature. First, we provide a warning that solution obtained in the continuum model are not necessarily indicative of solutions that would obtain in large finite economies. Second, by showing that CEEI is not the only assignment mechanisms that is both efficient and fair, our paper poses the question what mechanisms are both efficient and fair (or, nearly equivalently in large markets, efficient and incentive compatible)?[4] The study of these mechanisms is a topic for further research.

Let us stress that in many problems the qualitative properties of continuum economies do parallel those of large finite economies we are interested in. For instance, the asymptotic equivalence of Random Priority and Probabilistic Se-

[2]Thomson and Zhou (1993) have a similar characterization for assignment economies beyond unit demands. This result does not extend to our model because they only consider individual allocations belonging to the interior of the consumption space. In addition to Hylland and Zeckhauser, and Ashlagi and Shi, CEEI was studied by, among others, Azevedo and Budish (2013) who proved that CEEI becomes incentive compatible in large economies; Pycia (2011) who has shown that the CEEI assignment can be unboundedly more utilitarian-efficient than the best symmetric ordinal mechanism, and Hafalir and Miralles (2014) who have shown that under some conditions the CEEI assignment is utilitarian- and Rawlsian-optimal among all incentive-compatible assignment rules. Budish (2011) provided a deterministic approximation to CEEI, and Budish, Che, Kojima and Milgrom (2013) extended CEEI to multi-unit assignment problems. We study fairness in the standard sense of envy-free, see Foley (1967) and Kolm (1971).

[3]Zhou assumes that agents' utilities are monotone, quasiconcave and differentiable. His fairness concept is strict envy-freeness, which is equivalent to envy-freeness in our setting. In general, an allocation is strict envy-free if no agent envies the average bundle of a group of other agents. Zhou does not require agents to demand at most one object.

[4]We know from Miralles and Pycia (2014) that all such mechanisms can be described as competitive equilibria from some profile of incomes; the open question is how to assign the incomes in a way that is fair or incentive compatible.

rial mechanisms (Che and Kojima 2010) and the uniqueness of asymptotically ordinally efficient, symmetric, and strategy-proof mechanisms (Liu and Pycia 2013) obtain both in large finite and in continuum models.[5]

In more general economies, there is a rich literature shedding light on the benefits and shortcomings of working directly at limit economies. Roberts and Postlewaite (1976) find sequences of economies in which incentives to misreport preferences do not asymptotically vanish as the economy grows larger. Manelli (1991) constructs examples of sequences of increasingly large economies whose core cannot be decentralized via prices (a so-called core convergence failure), when preferences are not monotonic.[6] Serrano, Vohra and Volij (2001) show that, under asymmetric information, the core fails to converge to any sort of set of price equilibrium outcomes in replica economies.

2 Model

We study an economy with agents $i, j \in I \subset [0,1]$ and a finite, fixed set of indivisible objects $x, y \in X = \{1, 2, ..., |X|\}$. I is endowed with a measure λ,[7] and the total mass of I is 1. We allow, yet not impose, I to be finite with $|I|$ individuals, considered as $|I|$ atoms on the $[0,1]$ interval with mass $1/|I|$ each. In such a case we say that the economy is finite. Each object x is represented by a mass of identical copies (or capacity) $s_x \in (0,1)$. By $S = (s_x)_{x \in X}$ we denote the total supply of object copies in the economy. If agents have outside options, we treat them as objects in X; in particular, this implies that $\sum_{x \in X} s_x \geq 1$.

We assume that agents demand at most one copy of an object. We allow random assignments, and denote by $q^x(i) \in [0,1]$ the probability that agent i obtains a copy of object x. Agent i's random assignment $q(i) = (q^1(i), ..., q^{|X|}(i))$ is a probability distribution. The economy-wide assignment $q : I \to \Delta^{|X|-1}$ is feasible if $\int_I q(i) d\lambda \leq S$. Let $\mathcal{A}$ denote the set of economy-wide random assignments, and $\mathcal{F} \subset \mathcal{A}$ denote the set of feasible random assignments.

Agents are expected utility maximizers, and agent i's utility from random assignment $q(i)$ equals the scalar product $u_i(q(i)) = v(i) \cdot q(i)$ where $v(i) = (v^x(i))_{x \in X} \geq 0$ is the vector of agent i's von Neumann-Morgenstein valuations for objects $x \in X$. We assume that no agent is indifferent among all objects and that there is an object the agent strictly prefer to other objects.[8] This allows us to normalize valuation vectors so that each agent's highest valuation is 1 and

[5] See also Miralles (2008) for an exploration of the continuum model. Beyond the no-transfer model we study, the convergence results have been obtained by many authors. See, for instance, Gretsky, Ostroy and Zame (1992) for a study of core convergence as the economy converges to the atomless housing assignment model with transfers.

[6] See also Anderson and Zame (1997) who study core convergence when the set of goods is a continuum. They show that Edgeworth's conjecture that the core can be asymptotically decentralized through prices as the number of agents increase is not always true.

[7] When $I = [0,1]$, λ is the Lebesgue (uniform) measure.

[8] The assumption that no agent is indifferent among all objects is with no loss of generality. Such an agent would not ever envy other agents since she would be satiated by any probability bundle. Since she is satiated, it is easy to include her in any competitive equilibrium by endowing her with sufficient income.

lowest is 0. Let $\mathcal{V}$ be the space of all such normalized valuation vectors:

$$\mathcal{V} = \{(v^1, ..., v^{|X|}) \in [0,1]^{|X|} \mid \min\{v^1, ..., v^{|X|}\} = 0, \max\{v^1, ..., v^{|X|}\} = 1\}.$$

A valuation profile is a function $v : [0,1] \to \mathcal{V}$ that is measurable according with λ. An *assignment economy* is a tuple $E = (I, \lambda, X, S, v)$.

An economy is ε-*dense* if for any $v \in \mathcal{V}$ there is $i \in I$ with $\|v(i) - v\| < \varepsilon$.[9] An economy is *dense* if for every $\varepsilon > 0$ the economy is $\varepsilon-$dense. Clearly, no economy with finite I can be dense. A sequence of ε-dense economies where $\varepsilon \to 0$ converges to a dense economy. An economy has *full support* if for all $V \subset \mathcal{V}$ with $\dim(V) = \dim(\mathcal{V})$[10] we have $\lambda(V) > 0$. Obviously all fully supported economies are dense, yet the converse may not be true.

For an economy $E = (I, \lambda, X, S, v)$, a feasible random assignment $q^* \in \mathcal{F}$ is *Pareto efficient* (or, simply, *efficient*) if no other feasible random assignment $q \in \mathcal{F}$ is weakly preferred by all agents and strictly preferred by a positive mass of agents. A feasible random assignment $q^* \in \mathcal{F}$ is *envy-free* if for every pair of agents $i, j \in I$ we have $u_i(q^*(i)) \geq u_i(q^*(j))$.

A random assignment q^* and a price vector $p^* \in \mathbb{R}^{|X|}$ constitute a *competitive equilibrium* for a λ-measurable budget function $w^* : I \to \mathbb{R}$ if q^* is feasible, $p^* \cdot q^*(i) \leq w^*(i)$ for any $i \in I$, and $u_i(q(i)) > u_i(q^*(i)) \Longrightarrow p^* \cdot q(i) > w^*(i)$ for any $q \in \mathcal{A}$, and the complementary slackness condition obtains: for each good x either $p^{*x} = 0$ or $\int_I q^x(i) d\lambda = s_x$. In a *competitive equilibrium with equal incomes* (*CEEI*) we additionally require w^* to be a constant function.

Notice that when all prices in p^* are equal then every agent obtains sure assignment of her most preferred object. This case is straightforward, and in the sequel we focus on the case in which not all prices are equal; this allows us to normalize the prices and budgets so that the highest price is 1 and the lowest price is 0. Let $\mathcal{P}$ denote the set of all such normalized price vectors.

Sequences of economies and convergence

Let $t = 1, 2, ...$ A sequence of finite economies $E_t = (I_t, \lambda_t, X\ , S_t, v_t)$ is a *growing sequence of economies* if $\forall t$, $I_t \subsetneq I_{t+1}$ and $\forall i \in I_t$, $v_{t+1}(i) = v_t(i)$. A growing sequence of finite economies is a *converging sequence of economies* with limit $E = (I, \lambda, X\ , S, v)$ if $\forall t$, $I_t \subset I$ and $v_t(i) = v(i)$, and $S_t \to S$. In such a growing sequence we use $t(i) = \min\{t : i \in I_t\}$ for the moment in which an agent is incorporated into the sequence of economies. An assignment q *corresponds* to an economy $E = (I, \lambda, X\ , S, v)$ if it gives a lottery to each element of I. In a converging sequence of economies $E_t = (I_t, \lambda_t, X\ , S_t, v_t)$ with limit $E = (I, \lambda, X\ , S, v)$, a sequence of corresponding assignments q_t *payoff-converges* to q if for every $i \in I$ and for $t \geq t(i)$, $u(q_t(i)) \to u(q(i))$.

[9]We use the Euclidean norm although results do not depend on this particular choice of norm.

[10]The dimension of V is the maximum dimension along all convex subsets of V.

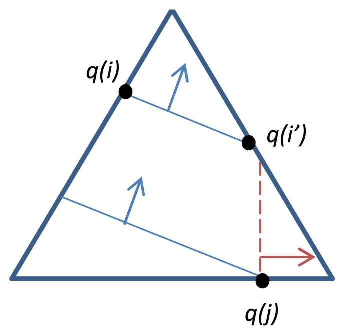

Figure 1: An efficient and envy-free assignment supported by a competitive equilibrium with income differences.

3 The Failure of Convergence

Is it true that as the number of agents grow and the economy becomes denser, every efficient and envy-free assignment are supported by a competitive equilibrium with arbitrarily similar incomes? This is indeed the case in the limit continuum economy with full support of preferences. As implied by Ashlagi and Shi (2014), in any full-support economy $E = (I, \lambda, X\ \ , S, v)$ with a continuum of agents every efficient and envy-free assignment q^* can be supported by a CEEI.[11]

However, this natural result fails in large finite economies. Let us start by illustrating that, not surprisingly, not every efficient and envy-free assignment can be supported by a CEEI in finite economies.

Figure 1 illustrates an assignment for the set of agents $\{i, i', j\}$ in the simplex of all probabilistic assignments of objects $\{x, y, z\}$. The simplex is drawn so that the top corner is the sure assignment of object x, the right corner represents the sure assignment of object y, and the left corners represents the sure assignment of object z. Individual assignments are $q(i) = (3/4, 0, 1/4)$, $q(i') = (1/2, 1/2, 0)$ and $q(j) = (0, 3/4, 1/4)$. Agents i and i' are both indifferent between each other's assignment: $v(i) = v(i') = (1, 1/2, 0)$ and the continuous lines reflect their indifference curves. Agent j's valuation vector is $v(j) = (1/2, 1, 0)$, and the dashed line represents her indifference curve. As usual, arrows indicate the direction towards which agents would be better-off. Notice that no agent envies another agent's assignment. This assignment is efficient because the price vector $p^* = (1, 1/2, 0)$ supports this assignment in an equilibrium. This price vector is in fact the unique equilibrium price vector supporting this assignment (in our normalization); but at these prices agents i and i' need budget $\frac{3}{4}$ to buy their

[11] Ashlagi and Shi (2014)'s Theorem 1 says that in a full-support economy every efficient, symmetric, and incentive-compatible mechanism can be supported by a CEEI as long as changes of reports by a measure zero of agents have no impact on the assignment of other agents. The results are related as under the latter assumption, envy-freeness is equivalent to a conjunction of symmetry and incentive compatibility in any full support continuum economy.

bundles while agent j must have the budget of exactly $\frac{3}{8}$. Hence this efficient and envy-free assignment cannot be implemented in CEEI.

This example leaves open the possibility that the characterization obtains not only in the limit economy but also becomes nearly true in large economies. Our main result addresses this possibility.

Theorem 1. ***(CEEI Convergence Failure)*** *There exists a converging sequence of ε_t-dense finite economies $E_t = (I_t, \lambda_t, X \ , S_t, v_t)$, with $|I_t| = t$, $\lambda_t(i) = 1/t \ \forall i \in I_t$ and $\varepsilon_t \to 0$, and a converging sequence of corresponding efficient and envy-free assignments q_t^* that* cannot *be supported by a corresponding sequence of equilibrium prices $p_t^* \in \mathcal{P}$ and income functions $w_t^* : I_t \to [0,1]$ such that $\max_{i,j \in I_t} [w_t^*(i) - w_t^*(j)] \to 0$.*

We prove the theorem by constructing a sequence of economies and assignments based on the example of Figure 1. At each step of the sequence, j is the agent who has the highest valuation for object x and prefers y over all other objects (including x). In ϵ-dense economies, j's utility difference between y and x is bounded by ϵ. We move $q(j)$ towards the sure assignment of object y in order to avoid envy. Yet $q\,(j)$ does not ever reach the limit sure assignment of object y. Hence j's income never reaches p^{*y}, which we keep equal to $\frac{1}{2}$ while we fix other agents' income at $\frac{3}{4}$.

Proof: Let $X = \{x, y, z\}$ and define $v_t = 1 - 1/4^t$ and $\varepsilon_t = 2\,(1 - v_t)$. We construct an ε_t-dense economy in which $\max\{v_t^x(i) : i \in I_t, v_t^y(i) = 1\} = v_t$ (recall that by our normalization $v_t^y(i) = 1$ whenever i ranks y at the top). Otherwise, we impose no constraints in the economy except that we want it to be ε_t-dense.

Consider assignments q_t^* defined so that agents who prefer y over all other goods (and thus who satisfy $v_t^y(i) = 1$) obtain a bundle with the set of goods $\theta_t = \{y, z\}$ with $q_t^y = \frac{1+v_t}{2}$ and $q_t^z = \frac{1-v_t}{2}$. Agents who prefer z over all other goods ($v_t^z(i) = 1$) obtain a sure copy of this object. Agents with $v_t^x(i) = 1$ and $v_t^y(i) > 1/2$ obtain a bundle of goods $\theta = \{x, y\}$ with $q_\theta^x = 1/2 = q_\theta^y$. Agents with $v_t^x(i) = 1$ and $v_t^y(i) < 1/2$ obtain a bundle of goods $\theta' = \{x, z\}$ with $q_{\theta'}^x = 3/4$ and $q_{\theta'}^z = 1/4$. We make sure that for every $t = 1, 2...$ there is at least one $i' \in I_t$ such that $v_t^x(i') = 1$, $v_t^y(i') = 1/2$ and $v_t^z(i') = 0$; this agent is indifferent between assignments θ and θ' and she obtains a non-degenerate lottery between them. This ensures that the normalized competitive equilibrium price vector that follows is unique. Furthermore, we ensure that the number of agents receiving each type of assignments and the supply vector are such that this assignment is feasible and there is no excess supply.

The assignments q_t^* are efficient and envy-free. Ffficiency is implied by the Hylland and Zeckhauser's (1979) First Welfare Theorem because there is a competitive equilibrium supporting it. The competitive equilibrium supporting q_t^* is constructed as follows. We set $p_t^* = v_t(i')$, that is, $p_t^{*x} = 1$, $p_t^{*y} = 1/2$ and $p_t^{*z} = 0$, $t = 1, 2, ...$ with incomes $w_t^*(i) = \frac{1+v_t}{4} < 1/2$ if i prefers y over all other goods, and $w_t(i) = 3/4$ otherwise.

The price vector is the unique (normalized) equilibrium price vector since

i''s assigned lottery is in the interior of the consumption space: linear utilities impose that i' is indifferent among all the probability bundles that cost the same in any competitive equilibrium. Given the uniqueness of p_t^*, all agents chosing either θ or θ' must have income $3/4$. We finally endow agents preferring y over all other objects with income income $\frac{1+v_t}{4}$ to make sure they purchase their assigned bundle. Therefore no other (normalized) prices and budgets can support assignments q_t^* at each $t = 1, 2, ...$

To check envy-freeness note that the richest agents in each equilibrium, those with income $3/4$, cannot envy any other agent's probability bundle (since it is also affordable with the highest income). Hence we just need to show that agents preferring y to other objects (that is, agents with $v_t^y(i) = 1$) do not envy the richest agents even though they have income $\frac{1+v_t}{4} < 3/4$. These agents obtain expected utility $\frac{1+v_t}{2} + \frac{1-v_t}{2} v_t^z(i)$. Their utility is thus weakly higher than $3/4 v_t^x(i) + 1/4 v_t^z(i)$ that bundle θ' gives, since $\frac{1+v_t}{2} > 3/4$ for every $t = 1, 2, ...$ and $v_t^x(i) < 1$; their utility is also weakly higher than $\frac{1+v_t^x(i)}{2}$ they would get from bundle θ (recall that $v_t^x(i) \leq v_t$ by definition).

We complete the proof by observing that $\max_{i,i' \in I_t} [w_t^*(i) - w_t^*(i')] > 1/4$ for every $t = 1, 2, ...$; thus income differences among agents do not asymptotically vanish in the equilibria we constructed. **QED**

Notice what happens in the limit economy to which the sequence E_t converges in our counterexample. In the limit economy, the agents who prefer y over other objects obtain a sure copy of y, and hence we can increase their budget (or income) without affecting the equilibrium; in particular, we can set their income to be equal to everyone else's. This is not possible at any finite economy, no matter how close to the continuum limit it is. This observation reconciles our construction with the work of Ashlagi and Shi (2014). : *The limit assignment of the sequence is a CEEI assignment; yet no element of the sequence is approximately CEEI.*

4 Conclusions

We show that the relation between large no-transfer assignment markets and their continuum-agent models is subtle. The problems are driven by the inherent failure of local non-satiation in such markets. By showing that the class of mechanisms that are fair and efficient (or symmetric, incentive-compatible, and efficient) in large finite markets is larger than CEEI, our note also poses the question what non-CEEI mechanisms satisfy these standard requirements.

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5 Appendix. Payoff Convegence

A natural question then is whether there is a different convergence notion that restores the validity of CEEI in large markets. Particularly, whether there exists a sequence of CEEI assignments for the *same* sequence of economies that converges to the same limit assignment. We would talk of *payoff convergence* of CEEI, a satisfactory result if one is concerned only about agents' payoffs. CEEI alone would constitute a good approximation to the set of payoffs obtained under efficient and envy-free assignments. Below we prove a weak version of payoff convergence. A sequence of CEEI assignments payoff-approaches any converging sequence of efficient and envy-free assignments *if the supply vector is slightly trembled* (and this tremble converges to zero as the economy grows large).

Remark 1 (CEEI Payoff Convergence under Trembling Supplies) Let $E_t = (I_t, \lambda_t, X\ , S_t, v_t)$, $t = 1, 2, ...$ be a converging sequence of economies with full-support limit $E = (I, \lambda, X\ , S, v)$, and let us have a sequence of corresponding efficient and envy-free assignments q_t^* that payoff-converges to q^*, an efficient and envy-free assignment corresponding to E. Then there is a parallel converging sequence of economies $\mathcal{E}_t = (I_t, \lambda_t, X\ , \Sigma_t, v_t)$ identical to E_t in all except for the supply vector, and with $\Sigma_t - S_t \to 0$, such that there is a corresponding sequence of CEEI assignments q_t^{ceei} for each $\mathcal{E}_t$ in which for every i and $t \geq t(i)$, $u(q_t^*(i)) - u(q_t^{ceei}(i)) \to 0$.

Proof: The assignment q^* is an efficient and envy-free assignment corresponding to E, a full-support economy. By Ashlagi and Shi (2014), it can be supported by a CEEI with prices p^* and the same budget w^* for every agent $i \in I$. Now we use a consistency property of competitive equilibria. For each t, let $q_t^{ceei} : I_t \to \Delta^{|X|-1}$ be equal to q^* with domain I_t: $q_t^{ceei}(i) = q^*(i)$ for every $i \in I_t$. Let $\Sigma_t = \int_{I_t} q_t^{ceei}(i) d\lambda_t + [S - \int_I q^*(i) d\lambda]$. (The latter component of the sum is the possible excess supply in the limit economy). It is clear that q_t^{ceei} is a CEEI assignment corresponding to $\mathcal{E}_t = (I_t, \lambda_t, X\ , \Sigma_t, v_t)$ with the same prices p^* and the same budget w^* for every agent $i \in I_t$. Moreover $\mathcal{E}_t \to E$, thus $\Sigma_t - S_t \to 0$. Since q_t^* payoff-converges to q^* and q_t^{ceei} is equal to q^* with domain I_t, we conclude that $u(q_t^*(i)) - u(q_t^{ceei}(i)) \to 0$. **QED**

A proper payoff convergence without trembling supplies needs that the set of payoff vectors obtained under CEEI be lower hemicontinuous in Σ_t, for every t. In that case, since $\Sigma_t - S_t \to 0$, for each sufficiently high value of t a CEEI under S_t would exist with payoffs close to those under Σ_t, which in turn approach those of the converging sequence of efficient and envy-free assignments. However, this condition does not always hold.

Remark 2 (**Discontinuity of Pseudomarket Equilibria**) There is a finite economy $E = (I, \lambda, X\ , S, v)$ such that the set of CEEI price and income vectors is not singleton nor lower hemicontinuous in S. Moreover, the corresponding set of payoff profiles does not have any of these properties either.

Proof: Let $X = \{x, y, z\}$, and let $I = \{i_1, i_2, i_3\}$ (each element with mass $1/3$). Supply is $S = (5/9, 2/9, 2/9)$. Valuations are $v(i_1) = (1, 0, 1 - \varepsilon)$, $v(i_2) = (1, 1 - \varepsilon, 0)$, $v(i_3) = (1, 1/2, 0)$, where ε is a small positive number.

For the easiness of exposition and only for this proof, we normalize all incomes to 1 instead of normalizing the highest price, which may now be higher than one. The lowest price is still normalized to zero. Uniqueness and lower hemicontinuity (as well as the lack of them) are preserved when we return to the original normalization.

Non-uniqueness.- The following are equilibrium price vectors: $p^* = (3/2, 3/4, 0)$, and $p' = (3/2, 0, 3/4)$. Corresponding assignments are $q^*(i_1) = q^*(i_3) = (2/3, 0, 1/3)$ and $q^*(i_2) = (1/3, 2/3, 0)$, under p^*, and $q'(i_1) = (1/3, 0, 2/3)$ and $q'(i_2) = q'(i_3) = (2/3, 1/3, 0)$, under p'. It is clear that both assignments integrate to S. It is also easy to check that agents obtain different payoffs depending on the equilibrium price vector. There are no other equilibria.

No lower hemicontinuity.- Consider the vector p^* and notice that i_3 is indifferent between his assignment and $q^*(i_2)$. Now let us marginally increase supply for object z, and the other objects marginally decrease their supplies in order to make overall supply add up to 1. *Any* valid marginal variation around p^* involves $p^z = 0$ due to our normalization. In order to increase demand for z, a marginal increase of p^x is necessary (for agents who spend their budgets on objects x and z only, higher price of x forces to buy a bundle with a higher share of z). Since supply for object y has been reduced and equilibrium requires lower demand of it, a marginal *decrease* of p^y is necessary (for agents who spend their budgets on objects x and y only, lower price of y allows to buy a bundle with a lower share of y). But then, the increase of p^x jointly with the decrease of p^y make i_3's optimal choice jump to a point close to $q^*(i_2)$, reducing her individual demand for object z from $1/3$ to zero. This breaks any possible equilibrium with prices around p^* when S is marginally modified in the way mentioned before. Indeed the equilibrium is now unique with prices around p', thus payoffs also non-marginally differ from those under S and p^*. **QED**

The main problem is that a pseudomarket equilibrium could just vanish, not only vary, with a small variation of supplies. Hence the tremble of supplies we use in Remark 1 cannot be left aside.